GRIMES

On Cauchy's Modulus Surfaces

Mathematics

A. M.

1913

ON CAUCHY'S MODULUS SURFACES

BY

RUBY MABEL GRIMES

A. B. Yankton College, 1911

THESIS

Submitted in Partial Fulfillment of the Requirements for the

Degree of

MASTER OF ARTS

IN MATHEMATICS

IN

THE GRADUATE SCHOOL

OF THE

UNIVERSITY OF ILLINOIS

1913

UNIVERSITY OF ILLINOIS

THE GRADUATE SCHOOL

June 7 1913

I HEREBY RECOMMEND THAT THE THESIS PREPARED UNDER MY SUPERVISION BY

Ruby Mabel Grimes.

ENTITLED On Cauchy's Modulus Surfaces

BE ACCEPTED AS FULFILLING THIS PART OF THE REQUIREMENTS FOR THE

DEGREE OF Master of arts.

Arnold Emch In Charge of Major Work

J Townsend Head of Department

Recommendation concurred in:

Committee
on
Final Examination

Table of Contents.

Introduction:

In a letter to Coriolis, 1837, Cauchy, defining x as an implicit function of t, determined by the solution of certain equations 1) $F(x) = 0$, in which t enters as a parameter; and defining the principal values of the parameter t as those for which equation (1) and its derivative have common roots; states the following theorems:

First.

Designate by t a real or imaginary variable. A real or imaginary function of t, represented by x, is developable into a convergent series, according to ascending powers of t, provided that the modulus of t remains below a value for which the function of x ceases to be finite or continuous.

Second.

Every root of an equation is generally developable according to ascending powers of a parameter contained in the equation, provided that the modulus of this parameter remains inferior to the moduli of all its principal values.

Third.

Let t be the parameter contained in the first member of equation (1). Provided that the modulus of this parameter remains inferior to the moduli of all its principal values, the distinct roots of equation (1) are all developable into convergent series according to ascending powers of t. Suppose, furthermore, that with the modulus of t increasing, one separates the various groups of roots of equation (1) in such a manner that at the origin the number of groups be equal to the number of distinct roots, and that later on, (as t increases), two groups coincide in the moment when two roots which belong respectively to those two groups become equal, for a given

modulus of t, corresponding to a certain principal value of the parameter. The number of groups of roots will be completely determined for each particular value of the modulus of t and equation (1) can be resolved into several others, of which each gives separately the various roots comprised in a single group.

He proved these theorems in an article which appeared in C.R. t IV. p.773 (22 Mai 1837): and which is reprinted in Cauchy's collected works, Vol.IV. Series I., pp. 48-60.

In the proofs which Cauchy gives he studies the system of curves

$$T = \left|\frac{\pi(x+y\sqrt{-1})}{\varpi(x+y\sqrt{-1})}\right|$$

for different values of the parameter T, which is obtained in the case where equation (1)

$$F(x) = 0$$

has the form (3)

$$\pi(z) + t\varpi(z) = 0.$$

For certain values of t this equation may have multiple roots, and it is these values of t that Cauchy calls principal values. For such values of t, the curve (2). has multiple points. If for such a value of t, equation (3) has k equal roots, then the corresponding curve (2) has k branches through the multiple point, which divide the full angle into 2k equal parts.

On page 60 Cauchy speaks of the surface which is obtained by considering in the equation (2) x, y, and T, as Cartesian Coordinates in space.

At the points corresponding to the roots of equation (2) the surface touches the xy plane, while to the poles correspond trombone shaped portions of the surface which extend to infinity. At the points corresponding to multiple roots, we have saddle points at

which the tangent planes are parallel to the xy plane, and cut the surface in multiple branches thru this point.

An account of Cauchy's investigation is given by A. Brill and M. Noether.*

As surfaces of this kind were apparently first studied by Cauchy we shall call them Cauchy's Modulus Surfaces.

Definition:

Let $W = f(z)$ be a function of the complex variable z, which, with the exception of points of a non-continuous set, is analytic in the neighborhood of all finite points of the plane.

Consider the modulus

$|W| = |f(z)|$, of f(z),

which may be written in the form

$$\zeta = +\sqrt{\phi^2(x,y) + \psi^2(x,y)}$$

or, $= \zeta = +\sqrt{u^2+v^2}$, $(W = u+iv)$,

if, $u = \phi(x,y)$, and $v = \psi(x,y)$

represent respectively the real and imaginary parts of f(z).

We may now consider three rectangular coordinate planes and designate the coordinates by x, y, ζ^2, or Z, so that the xy plane coincides with the complex plane.

$$Z = \phi^2(x,y) + \psi^2(x,y),$$

or $\quad Z = u^2 + v^2,$

then is the equation of a surface which may be called a Modulus Surface of Cauchy.

(2) Branch Points:

The property of the branches of the curve in case of multiple roots of (4) can be proved in the following simple manner:

* Jahresbericht der Deutschen Mathematiker-vereinigung Vol. 3. Die Entwicklung der Theorie der algebraischen Functionen in älterer und neuerer Zeit. pp. 187-188. Article 31.

Let f(z) and g(z) be polynomials in z so that

(5) $z' = \frac{f(z)}{g(z)}$

is an irreducible rational function. Assuming the degree of f(z) equal to n and greater than that of g(z), then for every value of z' there are n values of z which are the roots of equation

(6) $z' g(z) - f(z) = 0.$

We can write this equation in the form

* (7) $A_0 z^n + A_1 z^{n-1} + \text{------} + A_n = 0.$

In (7) the A's are polynomials in z' and the roots of (7) depend upon the particular values of z'. Suppose now that for some value z_0' of z', (7) has k equal roots, so that we have the expansion

(8) $z' - z_0' = (z - z_0)^k (a_0 + a_1 z + \text{------} + a_{n-k} z^{n-k})$

in which for

$z = z_0$

the second parenthesis reduces to a_0. Writing

$z - z_0' = u + iv$ and $z - z_0 = \rho e^{i\theta}$, there is

(9) $u + iv = \rho^k (\cos k\theta + i \sin k\theta), \phi(\rho, \theta),$

in which $\phi(\rho, \theta)$ for $z = z_0$ reduces to some constant, say $\rho_0 e^{i\theta_0}$.

In the neighborhood of z= zo we may therefore write

$u + iv = \rho_0 \rho^k \left[\cos(k\theta + \theta_0) + i \sin(k\theta + \theta_0) \right]$, and hence,

(10) $u = \rho_0 \rho^k \cos(k\theta + \theta_0)$

$v = \rho_0 \rho^k \sin(k\theta + \theta_0)$

To any direction $v/u = \mu$, thru z_0', correspond in the z- plane the directions defined by the relation

$\tan (k\theta + \theta_0) = \mu$

If θ_μ is the smallest angle for which $\tan \theta_\mu = \mu$, we may write $k\theta + \theta_0 = \theta_\mu + \lambda\pi$, so that

* Rendiconti del Circolo Matematice Di Palermo. Tomo xxxiv, Anno 1912, " On Conformal Rational Transformations in a Plane"

$$\theta = \frac{\theta_\mu - \theta_0}{k} + \frac{\lambda\pi}{k} \; ; \quad \lambda = 0,1,2,\text{-------------------}, k-1.$$

Any two consecutive branches intersect at an angle π/k. For a direction $v/u = -1/\mu$, perpendicular to the first,

$$\cot (k\theta + \theta_0) = -\mu$$

$$\tan (k\theta + \theta_0 + \pi/2) = \mu$$

$$k\theta + \theta_0 + \pi/2) = \theta_\mu + \lambda\pi$$

$$\Theta = \frac{\Theta_\mu - \Theta_0}{k} - \frac{\lambda\pi}{k} - \pi/2k \; ; \quad \lambda = 0,1,2,\text{------}(k-1).$$

i.e. the corresponding branches in the xy plane bisect the angles formed by the branches of the first curve.

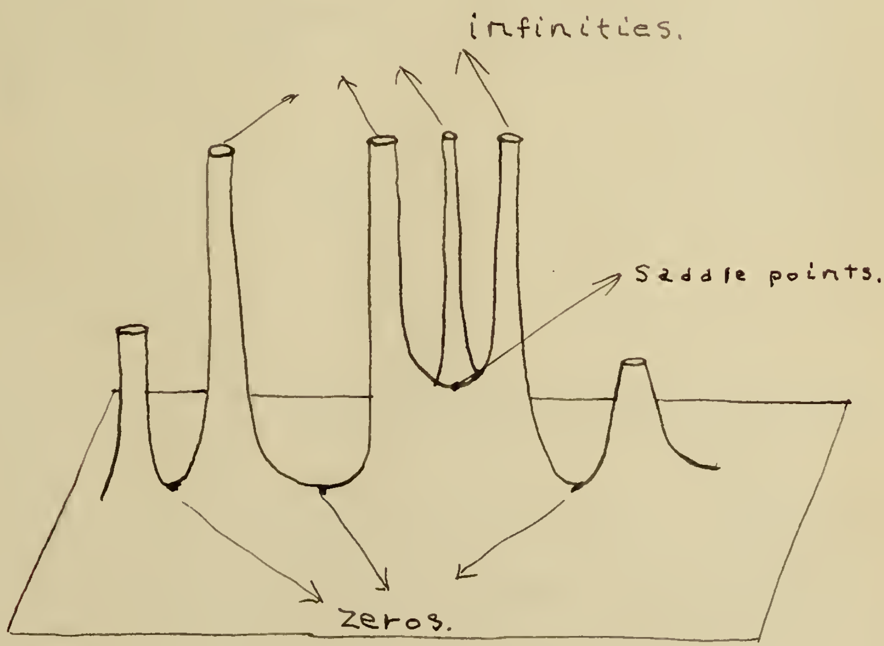


Fig. I.

Investigation of Simple Cases.

I. $W = az + b$.

The transformation $W = az + b$ is a linear transformation, consisting of similitude, a rotation, and a translation. a is a complex number and may be expressed as $Ae^{i\alpha}$. $z = \rho e^{i\theta}$. $\therefore W = Ae^{i\alpha} \cdot \rho e^{i\theta} = A\rho e^{i(\alpha+\theta)}$

= Similitude (coefficient A) + rotation thru the angle α.

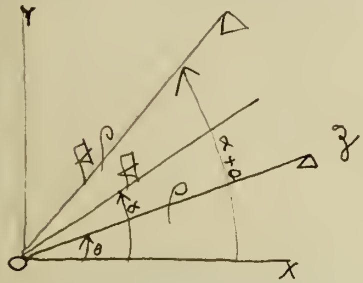


Fig. 2.

$W = z' + b$, translation.

$W = az + b$, similitude A + rotation α + translation b.

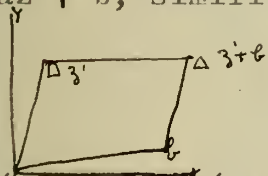


Fig. 3.

$$W = (a_1 + ia_2)(x + iy) + b_1 + ib_2$$

$$= a_1x + b_1 - a_2y + i(a_2x + a_1y + b_2).$$

$$Z = (a_1x + b_1 - a_2y)^2 + (a_2x + a_1y + b_2)^2$$

$$= a_1^2x^2 + a_2^2y^2 + a_2^2x^2 + a_1^2y^2 - 2a_1a_2xy + 2a_1a_2xy + 2a_1b_1x + 2a_2b_2x$$
$$+ 2a_1b_2y - 2a_2b_1y + b_1^2 + b_2^2$$

$$= (a_1^2 + a_2^2)x^2 + (a_1^2 + a_2^2)y^2 + 2(a_1b_1 + a_2b_2)x + 2(a_1b_2 - a_2b_1)y + b_1^2 + b_2^2.$$

Put $Z = k$, to determine the nature of the curves of intersection of planes parallel to the xy plane with the surface. Then we get

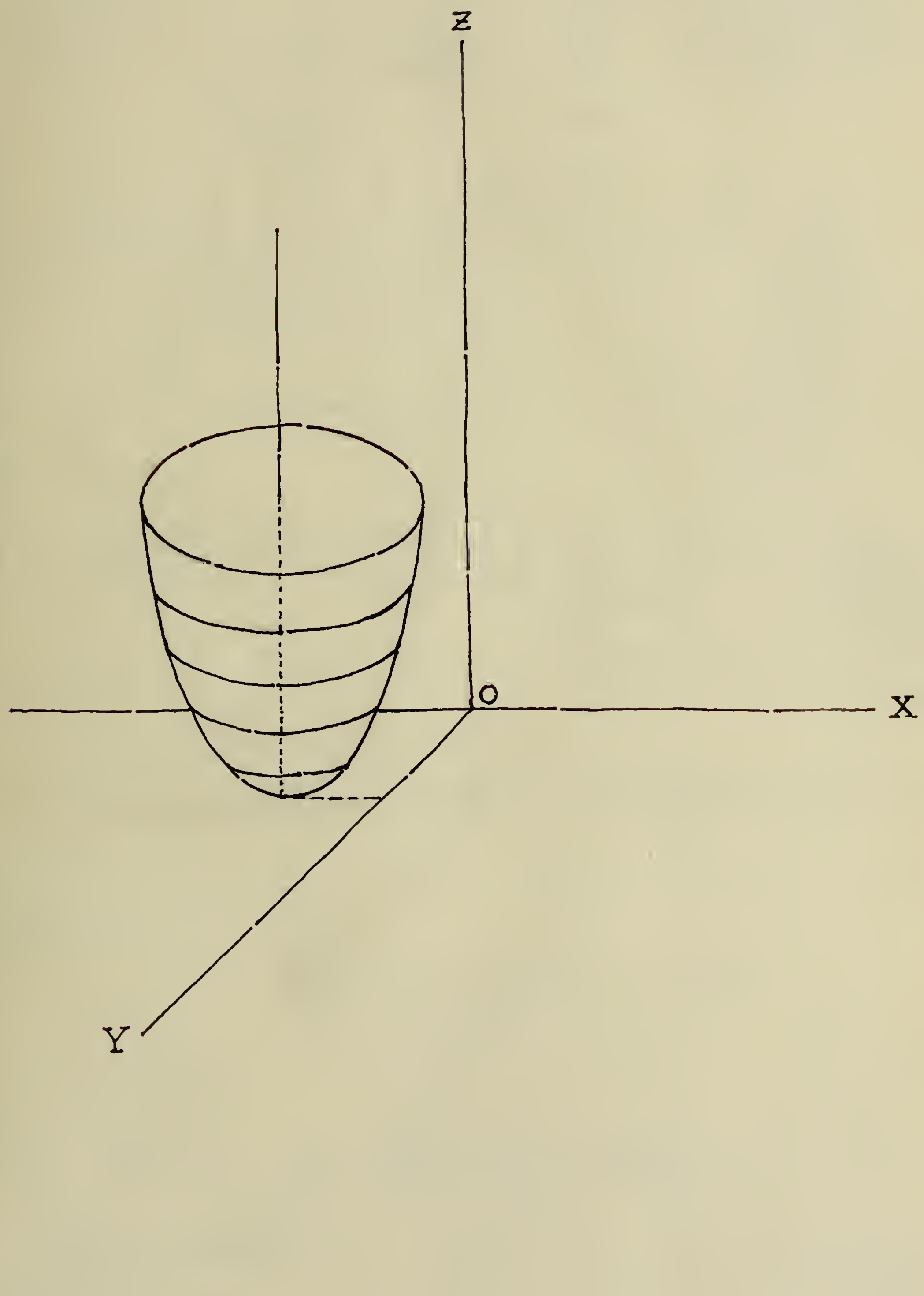
$$(x^2 + y^2) + \frac{2(a_1b_1 + a_2b_2)x}{a_1^2 + a_2^2} + \frac{2(a_1b_2 - a_2b_1)y}{a_1^2 + a_2^2} = K - \frac{b_1^2 + b_2^2}{a_1^2 + a_2^2}$$

Or, completing the square;

$$\left(x + \frac{a_1b_1 + a_2b_2}{a_1^2 + a_2^2}\right)^2 + \left(y + \frac{a_1b_2 - a_2b_1}{a_1^2 + a_2^2}\right)^2 = K - \frac{b_1^2 + b_2^2}{a_1^2 + a_2^2} + \frac{(a_1b_1 + a_2b_2)^2 + (a_1b_2 - a_2b_1)^2}{(a_1^2 + a_2^2)^2}$$

$$= K$$

FIG. IV



Therefore the sections made by planes parallel to the xy plane are circles and the surface is a paraboloid of revolution, with the axis of revolution perpendicular to the xy plane.

To find the nature of the sections made by planes parallel to the xz plane, let y = k and substitute this value for y in the equation of the surface. This then becomes

$$\left(x + \frac{a_1 b_1 + a_2 b_2}{a_1^2 + a_2^2}\right)^2 - Z = \left(k - \frac{a_1 b_2 - b_1 a_2}{a_1^2 + a_2^2}\right)^2$$

and the sections are seen to be parabolas. Semilarly the sections made by planes parallel to yz plane- found by putting x = k, are seen to be parabolas.

To find whether or not the vertex of the paraboloid lies in the xy plane, put Z = 0 and investigate the nature of the curve of intersection. This gives

$$\left(x + \frac{a_1 b_1 + a_2 b_2}{a^2 + a_2^2}\right)^2 + \left(y - \frac{a_1 b_2 - b_1 a_2}{a_1^2 + a_2^2}\right)^2 = 0$$

which is a null circle, or the point $\left(-\frac{a_1 b_1 + a_2 b_2}{a_1^2 + a_2^2} \cdot \frac{a_1 b_2 - b_1 a_2}{a_1^2 + a_2^2}\right)$.

Therefore the vertex of the paraboloid of revolution lies in the xy plane at $\left(-\frac{a_1 b_1 + a_2 b_2}{a_1^2 + a_2^2}, \frac{a_1 b_2 - b_1 a_2}{a_1^2 + a_2^2}\right)$

which in the superposed complex plane is equivalent to W = 0, or

Z = - b/a .

II. <u>The function W = 1/z</u>

$$u + iv = 1/x+iy = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x}{x^2 + y^2} \quad i \frac{-y}{x^2 + y^2}$$

$$\mathfrak{Z} = \sqrt{u^2 + v^2}$$

$$Z = u^2 + v^2$$

Or, substituting for u and v in terms of x and y,

$$Z = \frac{x^2}{(x^2 + y^2)^2} \frac{(-y)^2}{(x^2 + y^2)^2}$$

so that in this case,

(1) $$Z = \frac{1}{x^2 + y^2}$$

To find the sections of this surface made by planes parallel to

the xy plane, assign different constant values to Z and solve the resulting equation.

1) Let $z = 1$. Then $\frac{1}{x^2+y^2} = 1$; $x^2 + y^2 - 1 = 0$.

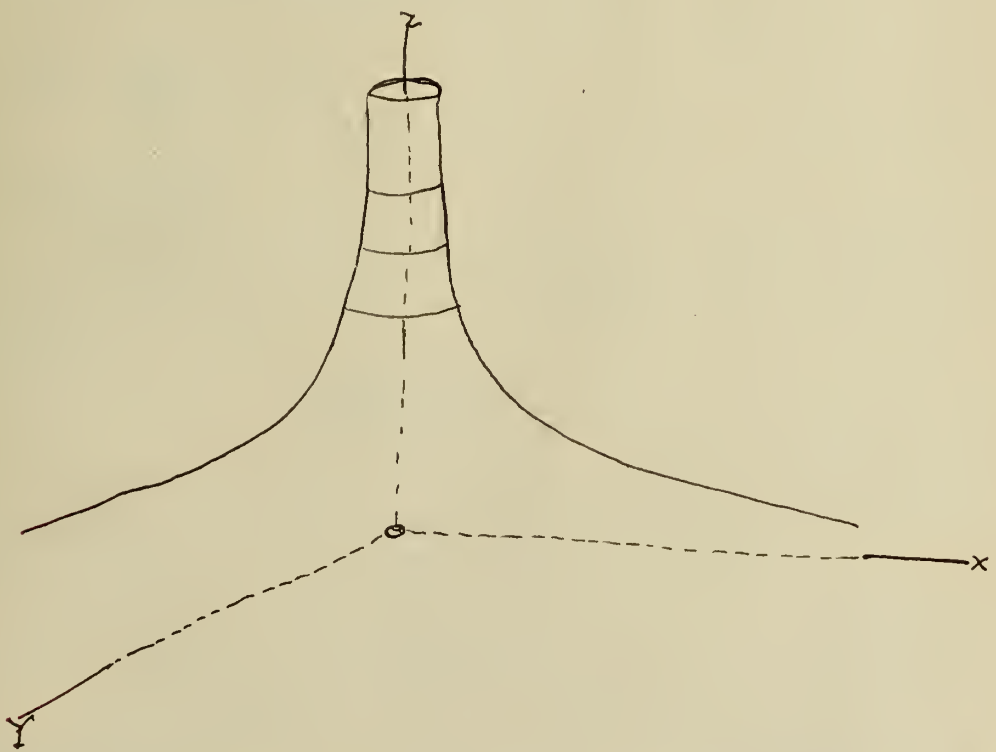
And we see that the section made by a plane parallel to the xy plane at a distance 1 above the xy plane is a circle of unit radius with its center at the origin.

2) Let $z = 2$. Then $\sqrt{u^2 + v^2} = 2$, $u^2 + v^2 = 4$ and $\frac{1}{x^2 + y^2} = 4$, $x^2 + y^2 = 1/4$; which is also a circle with its center at the origin and radius 1/2.

3) Let $z = 3$; Then the radius is 1/9.

4) In general for $z = k$, the sections are circles with their centers on the Z axis and radii equal to 1/k ; thus giving a surface such as is shown in

Figure V.



To find the sections made by the intersection of the surface and the other coordinate planes :

1. Consider the xz plane, $y = 0$. Substituting this value of y in the equation of the surface, we get $Z = 1/x^2$.

2. Similarly in the y plane, x = 0. Substituting, we get $Z = 1/y^2$, both giving cubic hyperbolas. See figure 6.

Now consider the section made by a plane parallel to the xZ plane, say y = 1. Substituting this value we get $\frac{1}{x^2+1} = Z$. Or, writing this in the implicit form it becomes $x^2z + z - 1 = 0$. To find the asymptotes tothis curve, we notice that it is a cubic equation, lacking the term x^3, and we may express it as

$$0.x^3 + z\,x^2 + z - 1 = 0.$$

Then since the coefficient of the highest power of x = 0, the coefficient of the next highest power of x, equated to zero gives the equation of the asymptote. Hence,

Z = 0 = Asymptote.

To find the double points of this curve we have the conditions that

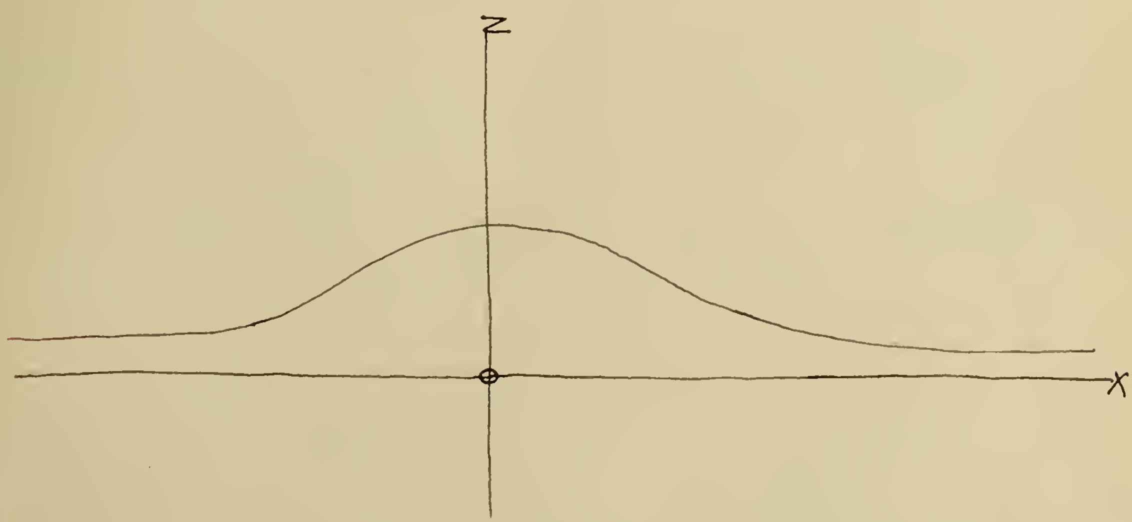
$$\frac{\partial f}{\partial z} = 2xz = \frac{2x}{x^2+1} = 0$$

and

$$\frac{\partial f}{\partial z} = x^2+1 = 0.$$

This is not possible for any value of x, so that the curve has no singular point.

Figure VI.



To find the points of inflexion we have $\frac{d^2z}{dx^2} = 0$ as a necessary and sufficient condition.

$$\frac{1}{x^2+1} = z$$

$$\frac{dz}{dx} = \frac{-2x}{(x^2+1)^2}$$

$$\frac{d^2z}{dx^2} = \frac{-2(x^2+1)^2 + 2x.2(x^2+1)\,2x}{(x^2+1)^4}$$

$$\frac{d^2z}{dx^2} = \frac{-2(x^4+2x^2+1)+8x^4+8x^2}{x^8+8x^6+6x^4+4x^2+1} = 0$$

Or $-2x^4-4x^2-2+8x^4+8x^2=0$

$$6x^4+4x^2-2=0$$

$$3x^4+2x^2-1=0$$

$$x^2 = \frac{-2\pm\sqrt{4+12}}{6} = \frac{-2\pm 4}{6} = 1/3 \text{ or } -1.$$

$\therefore x = \pm 1/3\sqrt{3}$ or $\pm i$

$$z = \frac{1}{x^2+1},\ = \frac{1}{1/3+1} = 3/4 \text{ or } \frac{1}{-1+1} = \infty$$

Therefore ($1/3\sqrt{3}$, $3/4$) and ($-1/3\sqrt{3}$, $3/4$) are the real points of inflexion of the curve.

Now let us consider the intersection of the given surface and a plane oblique to the xy - plane.

$ax+by+cz+d = 0 \equiv$ plane.

$Z = \frac{1}{x^2+y^2} \equiv$ surface.

Solving, we get as the equation of the projection of the curve of intersection, $ax+by+\frac{c}{x^2+y^2}+d = 0$, or

$(x^2+y^2)(ax+by+d)+c = 0$, which clearly is a circular cubic, with $ax+by+d = 0$, as a parallel to the asymptote.

1). $(x^2+y^2)(ax+by+d)+c = 0$.

2). Let $y = mx+k$. Then substituting in (3)

$ax^3+bx^2y+axy^2+by^3+dx^2+dy^2+c = 0$, we get

$ax^3+bx^2(mx+k)+ax(m^2x^2+2kmx+k^2)+b(m^3x^3+3km^2x^2+3k^2mx+k^3)$
$+dx^2+d(m^2x^2+2kmx+k^2)+c = 0.$

Collecting terms

$$x^3(a + bm + am^2 + bm^3) + x^2(bk + 2\,akm + 3bkm^2 + d + dm^2) + \text{----} = 0.$$

The conditions for two infinite equal roots are

(4) $a + bm + am^2 + bm^3 = 0$

(5) $(a + bm)(m^2 + 1) = 0,$

$m = -a/b$, or $\pm\, i$.

Substituting the real value in the coefficient of x^2 we get

$$bk + 2ak(-a/b) + 3bk(a^2/b^2) + d + d(a^2/b^2) = 0$$

Clearing of fractions,

$$b^3k - 2a^2kb + 3a^2bk + d + a^2d = 0$$

$$k(b^3 - 2a^2b + 3a^2b) + d + a^2d = 0$$

$$k = \frac{-d(a^2 + 1)}{b^3 + a^2b}$$

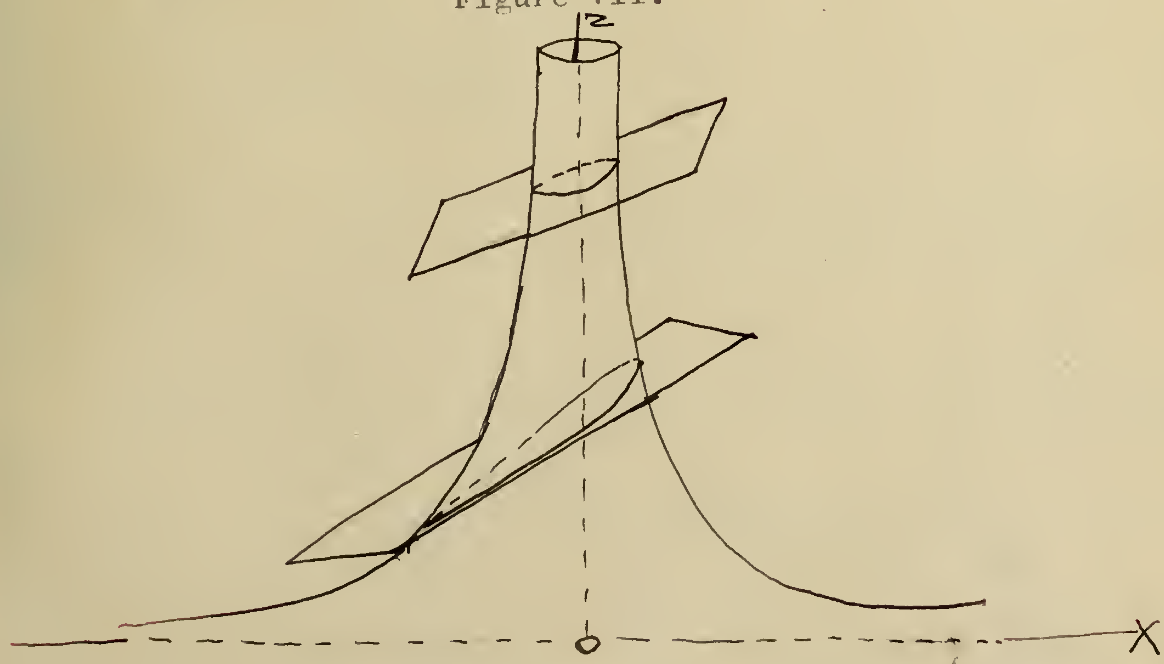
Thus $y = \frac{-a}{b}x - \frac{d(a^2 + 1)}{b^3 + a^2b}$ is equation of asymptote, or simplifying;

$$y = \frac{-a}{b}x - \frac{d(a^2 + 1)}{b^3 + a^2b}$$

$$b^3y + a^2by = -abx^2 - ax^3 - a^2d - d$$

(6) $(ab^2 + a^3)x + (a^2b + b^3)y + d(a^2 + 1) = 0$

Figure VII.



The general equation of a tangent plane to any surface at the point (x_1, y_1, z_1) is $(x - x_1)\frac{\partial f}{\partial x} + (y - y_1)\frac{\partial f}{\partial y} + (z - z_1)\frac{\partial f}{\partial z} = 0.$

Where the surface is represented by

$f(x,y,z) = 0$

$f(x,y,z) \equiv (x^2 + y^2)z - 1 = 0$

$\frac{\partial f}{\partial x} = 2xz = \frac{2x}{x^2 + y^2}$

$\frac{\partial f}{\partial y} = 2yz = \frac{2y}{x^2 + y^2}$

$\frac{\partial f}{\partial z} = x^2 + y^2$

Substituting in the general equation we get

$(x-x_1)\frac{2x_1}{x^2 + y^2} + \frac{2y_1}{x_1^2+y_1^2}(y - y_1)\ (z - z_1)(x_1^2 + y_1^2) = 0$

If we take as the point of tangency the point (x_1, o, z_1), this equation becomes

$\frac{2(x - x_1)}{x_1} + (Z - Z_1)x_1^2 = 0$ or,

$2(x - x_1) + x_1^3(Z - Z_1) = 0$ Tangent plane at (x_1, o, z_1)

Solving this equation simultaneously with that of the surface, we get,

$2(x - x_1) + x_1^3\left(\frac{1}{x^2 + y^2} - \frac{1}{x_1^2}\right) = 0$

(8) Or $2(x - x_1)(x^2 + y^2) + x_1(x_1^2 - x^2 - y^2) = 0$

And this gives the projection of the curve of intersection of the surface and the tangent plane at (x_1, o, Z_1) upon the xy plane. It a is a circular cubic.

To find the double points of this curve the conditions, $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$, must hold with (8).

1). $f(x,y) = 2(x - x_1)(x^2 + y^2) + x_1(x_1^2 - x^2 - y^2) = 0$

2). $\frac{\partial f}{\partial x} = 6x^2 - 6x_1 x + 2y^2 = 0$

3). $\frac{\partial f}{\partial y} = 4xy - 6x_1 y = 0$, $y(4x - 6x_1) = 0$, $y = 0$, $4x - 6x_1 = 0$.

Substituting $y = 0$ in (2) we find $x = x$ .This point $(x\ o)$ is on the locus and is evidently a real double point. The other common solution of 2) and 3) does not satisfy 1) and is therefore not a point of the locus.

To find the slope of the tangent to (1) at the double point (x_1, o).

$$\frac{dy}{dx} = -\frac{\left[\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x \partial y} \cdot \frac{dy}{dx}\right]}{\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial^2 f}{\partial y^2} \cdot \frac{dy}{dx}}$$

$$\frac{\partial^2 f}{\partial x \partial y} \frac{dy}{dx} + \frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dx}\right)^2 = -\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial x \partial y} \frac{dy}{dx} \quad \text{Or,}$$

$$\frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dx}\right)^2 + \frac{2\partial^2 f}{\partial x \partial y} \cdot \frac{dy}{dx} + \frac{\partial^2 f}{\partial x^2} = 0.$$

$$\therefore \frac{dy}{dx} = \frac{-\frac{\partial^2 f}{\partial x \partial y} \pm \sqrt{\left[\frac{\partial^2 f}{\partial x \partial y}\right]^2 - \frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2}}}{\frac{\partial^2 f}{\partial y^2}}$$

Substituting the coordinates of the double point in this equation after putting for

$$\frac{\partial f}{\partial x}, \quad 6x^2 + 2y^2 - 6x_1 x$$

$$\frac{\partial^2 f}{\partial x^2} = 12x - 6x_1$$

$$\frac{\partial f}{\partial y} = 4xy - 6x_1 y$$

$$\frac{\partial^2 f}{\partial y^2} = 4x - 6x_1$$

$$\frac{\partial^2 f}{\partial x \partial y} = 4y, \qquad \text{we get}$$

$$\frac{dy}{dx} = \frac{-4y \pm \sqrt{16y^2 - (12x-6x_1)(4x-6x_1)}}{4x - 6x_1}$$

$$= \frac{\pm\sqrt{-3(4x_1^2 - 8x_1^2\ \ 3x_1^2)}}{-x_1}$$

$$= \frac{\pm\sqrt{3x_1^2}}{-x_1} = \frac{\pm x_1\sqrt{3}}{-x_1} = \mp\sqrt{3}$$

As this value is independent of the location of the point of tangency in the xz - plane, <u>the two branches of the curve through the double point in the xy - projection intersect at a constant angle.</u>

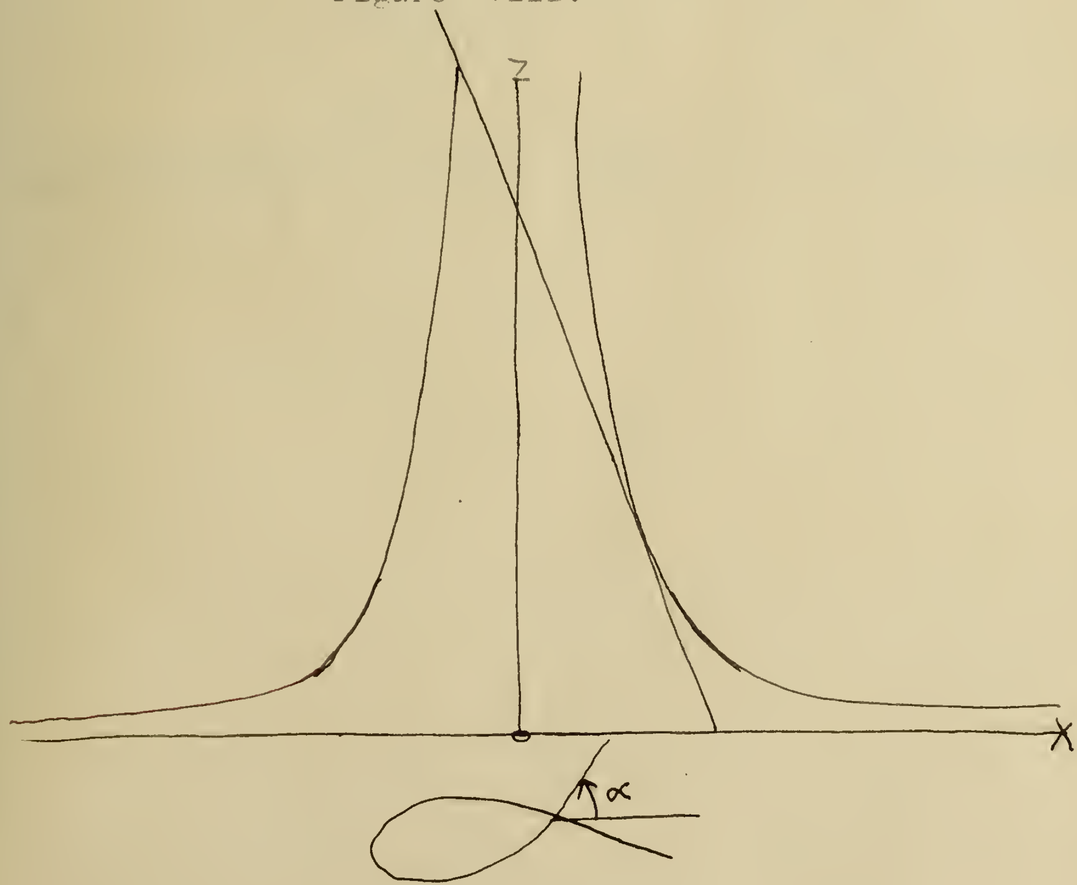
To find the trigonometric tangent of the angle between the two geometric tangents at the double point.

$$\operatorname{Tan} \theta = \frac{l_1 - l_1}{1 - l_1 l_2} = \frac{\sqrt{3} + \sqrt{3}}{1 - 3} = \frac{2\sqrt{3}}{-2} = -\sqrt{3}. \therefore \theta = 120^\circ$$

Hence the

<u>Theorem:</u> <u>Every tangent-plane cuts the Cauchy surface for $W = 1/z$ in a curve, whose xy projection is a circular cubic with a real double point, such that the branches of the cubic through the double point intersect at a constant angle of 120°.</u>

Figure VIII.



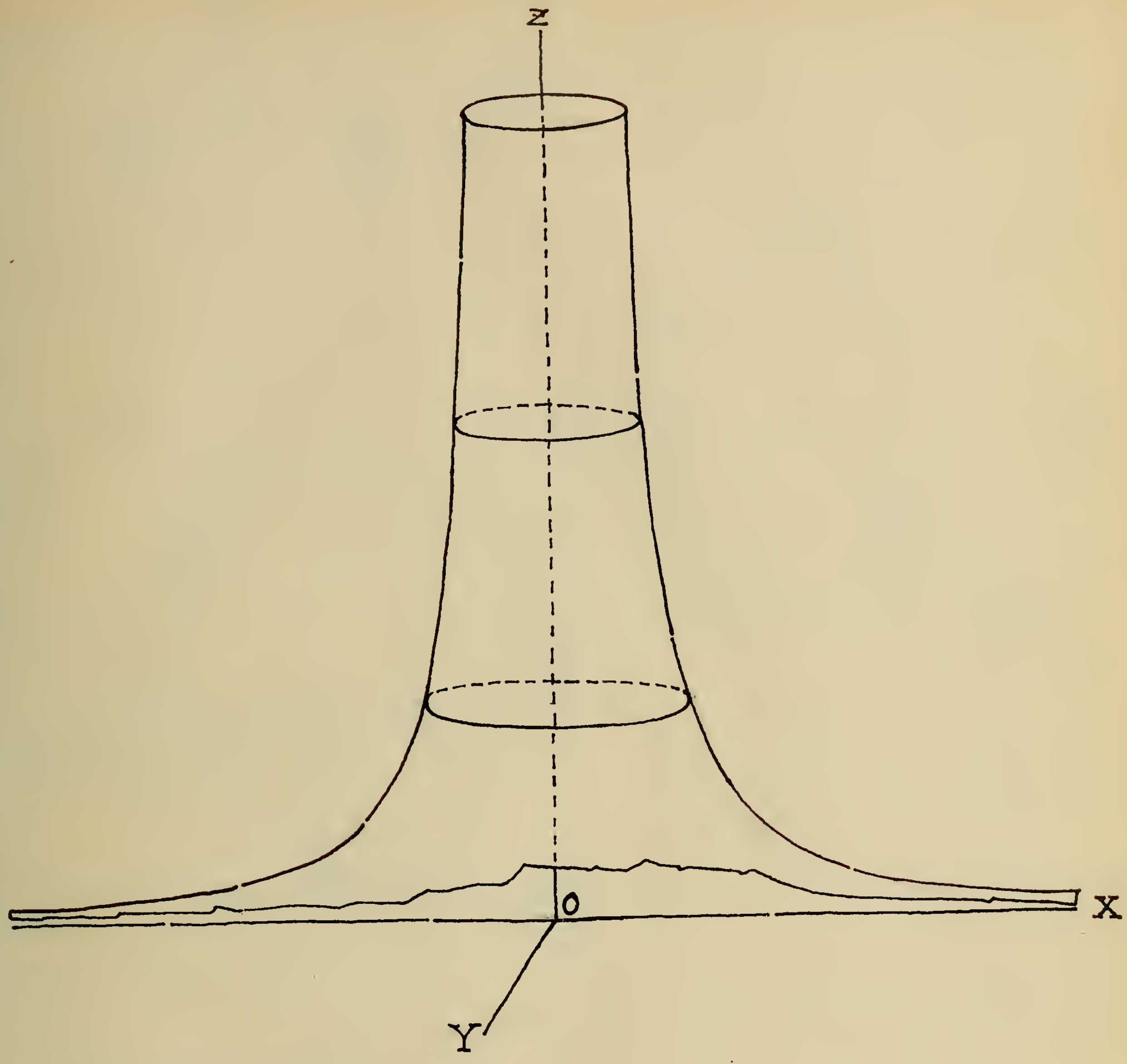


Fig. IX.

III. Consider the transformation

(1) $W = \dfrac{az+b}{cz+d} = \dfrac{a}{c}\,\dfrac{z-(-b/a)}{z-(-d/c)}$

$W = 0$ when $z = -b/a$

$W = \infty$ when $z = -d/c$

$W = a/c \;\dfrac{z-z_0}{z-z_\infty}$, where $z_0 = -b/a$, $z_\infty = -d/c$.

The factor a/c involves similitude and rotation. $W' = kz$, $k = Ke^{i\theta}$, so that as far as the configuration is concerned, nothing is lost in generality by restricting the investigation to the transformation $W = \dfrac{z-z_0}{z-z_\infty}$, or $W = \dfrac{z-\alpha}{z-\beta}$

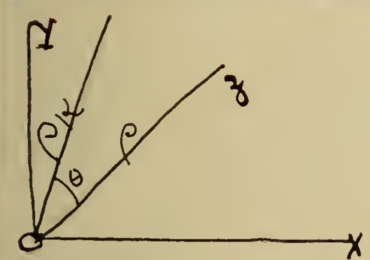


Fig I.

$W = \dfrac{x+iy-\alpha_1-i\alpha_2}{x-iy-\beta_1,-i\beta_2} = u+iV.$

As $Z = U^2+V^2$, it is found that

$U+iV = \dfrac{[x+iy-\alpha_1-i\alpha_2][(x-\beta_1)-i(y-\beta_2)]}{[(x-\beta_1)+i(y-\beta_2)][(x-\beta_1)-i(y-\beta_2)]}$

Or $x^2 + ixy - \alpha_1 x - i\alpha_2 x - \beta_1 x - i\beta_1 y + \alpha_1\beta_1 + i\alpha_2\beta_1 - ixy + y^2 - i\alpha_1 y + \alpha_2 y$

$\dfrac{+\, i\beta_2 x - \beta_2 y - i\alpha_1\beta_2 + \alpha_2\beta_2}{(x-\beta_1)^2+(y-\beta_2)^2}$

And $U = \dfrac{(x^2-\alpha_1 x+\beta_1 x+\alpha_1\beta_1+y^2+\alpha_2 y-\beta_2 y+\alpha_2\beta_2)}{(x-\beta_1)^2+(y-\beta_2)^2}$

$V = \dfrac{(\beta_2 x-\alpha_2 x-\beta_1 y+\alpha_2\beta_1-\alpha_1 y-\alpha_1\beta_2)}{(x-\beta_1)^2+(y-\beta_2)^2}$

It is always possible to determine a transformation consisting of a translation, rotation and similitude so that α and β are transformed into $+1$ and -1 respectively. For this purpose take the general form of the equation

$\zeta' = \dfrac{a\zeta+b}{c\zeta+d}$, so that $c\zeta\zeta' + d\zeta' - a\zeta - b = 0$

Suppose $\zeta = \alpha$ and $\zeta' = 1$. Then the equation becomes $c\alpha - a\zeta + d - b = 0$

Again suppose $\zeta = \beta$, $\zeta' = -1$. The equation then becomes $-c\beta - a\beta - d - b = 0$

$$a-c = \frac{d-b}{\alpha}\ , \text{ and } a-c = -\frac{d-b}{\beta}$$

$$a = \frac{1}{2}\left\{\frac{d-b}{\alpha} - \frac{d-b}{\beta}\right\}$$

$$c = \frac{1}{2}\left\{-\frac{d-b}{\beta} - \frac{d-b}{\alpha}\right\}$$

$$\therefore \zeta' = \frac{\frac{1}{2}\left\{\frac{d-b}{\alpha} - \frac{d-b}{\beta}\right\}\zeta + b}{-\frac{1}{2}\left\{\frac{(d-b)}{\alpha} + \frac{(d+b)}{\beta}\right\}\zeta + d}$$

$$\text{And } \zeta' = \frac{\left\{\beta(d-b) - \alpha(d+b)\right\}\zeta + 2\alpha\beta b}{-\left\{\beta(d-b) + \alpha(d+b)\right\}\zeta + 2\alpha\beta d}$$

Replacing in $W = \frac{z-\alpha}{z-\beta}$, z in terms of ζ' as extracted from this equation and writing afterward W for ζ' the transformation becomes

(2) $W = \frac{z-1}{z+1}$.

According to the general theory, page 5, the corresponding Cauchy surface of the function $\frac{z-1}{z+1}$, with z=1 as a zero and z = -1 as a pole, touches the xy - plane at (0,1) and has a trombone-shaped part extending in the direction of the line (y= 0, x = -1). To write the equation of the surface explicitly in Cartesian coordinates we have:

$$W = \frac{z-i}{z+1} = \frac{x+iy-1}{x+iy+1} = \frac{x-1+iy}{x+1+iy}\cdot\frac{x+1-iy}{x+1-iy} = \frac{x^2+y^2-1+2iy}{x^2+2x+1+y^2} = U + iv$$

$$\therefore U = \frac{x^2+y^2-1}{x^2+2x+1+y^2}\ , \text{ and } V = \frac{2y}{x^2+2x+1+y^2}$$

Hence for the equation of the surface

$$Z = U^2 + V^2 = \frac{(x^2+y^2-1)^2+4y^2}{(x^2+2x+1+y^2)^2}, \text{ or}$$

$$Z = \frac{x^4+y^4+1-2x^2-2y^2+2x^2y^2+4y^2}{x^4+4x^2+1 \quad +y^4+4x^3+2x^2+2x^2y^2+4x+4xy^2+2y^2}$$

Reducing the general equation of the surface we have:

$$Z = \frac{(x^2+y^2-1)^2+4y^2}{\left[(x+1)^2+y^2\right]^2} = \frac{x^4+y^4+1-2x^2-2y^2+2x^2y^2+4y^2}{\left[(x+1)^2+y^2\right]^2}$$

$$Z\left[(x+1)^2+y^2\right]^2 = \left[(x+1)^2+y^2\right]\left[(x-1)^2+y^2\right]$$
$$= \left[(x^2+y^2)+2x+1\right]\left[(x^2+y^2)-2x+1\right]$$
$$= (x^2+y^2)^2+2(x^2+y^2)+1-4x^2$$
$$= x^4+2x^2y^2+y^4+2x^2+2y^2+1-4x^2$$
$$= x^4+y^4+2x^2y^2-2x^2+2y^2+1$$

$(x+1)^2+y^2=0$ represents infinitely small circle at infinite distanc

$$Z\left[(x+1)^2+y^2\right]=\left[(x-1)^2+y^2\right]$$
$$(3)\quad \underline{Z\left[(x+1)^2+y^2\right]-\left[(x-1)^2+y^2\right]=0}$$

This is the equation of surface in its reduced form.

To find the nature of the sections made by the intersection of this surface and planes parallel to the xy - plane. Put $Z = k$, and we get

$k = \dfrac{(x-1)^2+y^2}{(x+1)^2+y^2}$ If $k = 2$, we have

$$2x^2+4x+2+2y^2 = x^2-2x+1+y^2$$
Or, $x^2+6x+1+y^2=0$
$$x^2+6x+9+y^2=8$$
$$(x+3)^2+y^2=8.$$

Therefore the section is a circle, of radius 8 and center at (-3,0).

Let $Z = 1/8$. Then we have

$x^2+2x+1+y^2=8x^2-16x+8+8y^2$, or,
$$-7x^2+18x-7-7y^2=0$$
$$x^2-\frac{18}{7}x+1+y=0$$
$$\left[x^2-\frac{18}{7}x+\left(\frac{18}{14}\right)^2\right]+y^2=-1+\left(\frac{18}{14}\right)^2$$
$$\left(x^2-\frac{18}{7}x+\frac{81}{49}\right)+y^2=\frac{4}{49}$$
$$(x-9/7)^2+y^2=\frac{4}{49}.$$

Therefore the section is a circle with radius 2/7 and center at

(9/7, 0). Similarly, putting $Z = 1/4$, we get a circle with radius 4/3 and center at (5/3, 0). So also with $Z = 3/4$, the section is a circle, with center at (7, 0) and radius $= \sqrt{48}$.

For the general case then, if $Z = k$,

$$kx^2 + 2kx + k + ky^2 - x^2 + 2x - 1 - y^2 = 0$$

$$(k-1)x^2 + 2(k+1)x + (k-1)y^2 + (k-1) = 0$$

$$\left\{x^2 + 2\left(\frac{k+1}{k-1}\right)x + y^2 + 1 = 0.\right.$$

$$\left[x^2 + 2\left(\frac{k+1}{k-1}\right)x + \left(\frac{k+1}{k-1}\right)^2 + y^2 = \left(\frac{k+1}{k-1}\right)^2 + 1\right.$$

(4) $\left[x + \frac{k+1}{k-1}\right]^2 + y^2 = \frac{4k}{(k-1)^2}$. Therefore the sections are circles with centers at $(-\frac{k+1}{k-1}, 0)$ and radii $\frac{2\sqrt{k}}{k-1}$.

Considering the projection on the xy plane, of all circles on the surface, we have a pencil with the imaginary base points, (0,i) and (0, -i). For

$$Z\left[(x+1)^2 + y^2\right] - \left[(x-1)^2 + y^2\right] = 0$$

The intersection of $(x+1)^2 + y^2 = 0$

and $(x-1)^2 + y^2 = 0$

are (0,i)(0,-i).

$$x^2 + y^2 + 2x + 1 = 0$$

$$x^2 + y^2 - 2x + 1 = 0$$

$$4x = 0$$

$$x = 0$$

$$y^2 = -1$$

$$y = \pm i$$

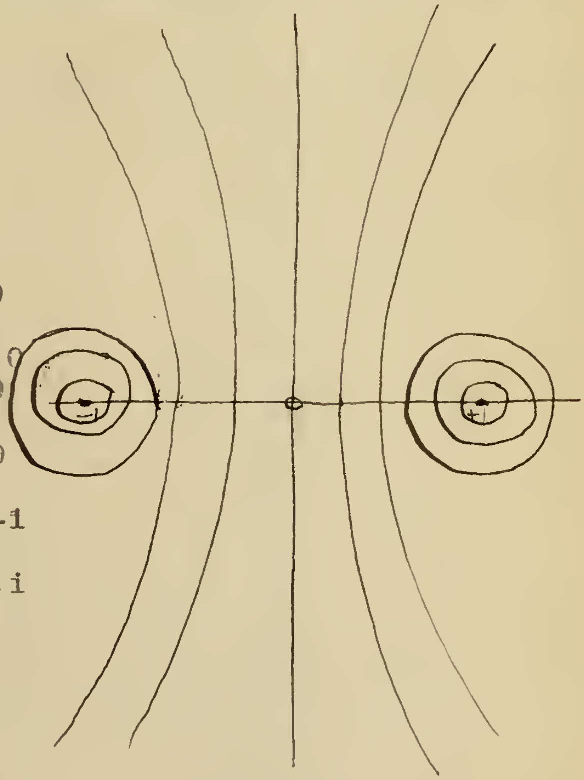


Fig. XI.

This is shown by figure 10. It follows, however, directly from the transformation $W = \frac{z-1}{z+1}$. For, $\left|\frac{z-1}{z+1}\right| = |W| = \sqrt{Z} = \text{constant}$, z moves on a circle k of the pencil with +1 and -1 as limiting points.

For arg.(z-1) - arg(z+1) = const. z moves on a circle of the conjugate pencil as is shown in Figure 11.

Table of values for different values of Z = k, for radius and distance of center of circle at distance k from xy plane.

Z K	center $\frac{-k-1}{k-1}$	radii $2\sqrt{k}/k-1$
0.1	1.22	.688
0.2	1.5	1.1
0.3	1.85	1.5
0.4	2.33	2.1
0.5	3.	2.8
0.6	4.	3.8
0.7	5.66	5.5
0.8	9.	8.9
0.9	19.	18.9
1.1	- 21.	20.876
1.2	- 11.	10.954
1.3	- 7.66	7.60
1.4	- 6.	5.916
1.5	- 5.	4.898
1.6	4.33	4.213
1.7	3.85	3.725
1.8	3.5	3.354
1.9	3.22	3.063
2	3.	2.828
3	2.	1.7321
4	1.66	1.333
5	1.5	1.118
6	1.4	.979

7	1.33	.8819
8	1.28	.8081
9	1.25	.75
10	1.22	.70

$\frac{-(z+1)}{z-1} = x$, $-z-1 = xz-x$; $xz-x+z+1 = 0 \equiv$ equation of line of centers of circles, and this we see is an equation of an equilateral hyperbola.

To determine the nature of the curves of intersection of the surface and the coordinate planes. Put first, $x = 0$, and we get

$$\frac{y^4+1+2y^2}{(y^2+1)^2} = Z = 1$$

as the section made by the yz plane. Therefore the line ($Z=-1, x=0$) lies in the surface.

To find the cross section of xz plane with the surface, put $y = 0$, and we have

(5) $Z(x+1)^2-(x-1)^2= 0$, or $Z = \left(\frac{x-1}{x+1}\right)^2$, a cubic.

To determine the asymptotes to this curve, let $Z = mx+b$, be the equation of the asymptote. Then

$$mx+b = \left(\frac{x-1}{x+1}\right)^2 \text{ or}$$

$(mx+b)(x+1)^2 = (x-1)^2$ and

$mx^3+ 2mx^2+mx+bx^2+2bx+b=x^2-2x+1$

$mx^3+(2m+b-1)x^2+(m+2b+2)x+b-1=0$

$m=0$, $2m+b-1=0$, $\therefore b=0$.

The equation of the asymptote is therefore $Z = 1$.

To determine the points of inflexion of the cubic $Z = \left(\frac{x-1}{x+1}\right)^2$, we have from

$$\frac{dz}{dx} = \frac{(2x-2)(x+1)^2-(x^2-2x+1)2(x+1)}{(x-1)^4} = \frac{4(x-1)}{(x+1)^3}$$

the condition

$$\frac{d^2z}{dx^2} = \frac{4(x+1)^3 - 4(x-1)(x+1)^2 \cdot 3}{(x+1)^6} = \frac{-8(x-2)}{(x+1)^4}.$$

This becomes 0 if $8x-16 = 0$, or $x = 2$, therefore $x=2$, $y=0$, is a point of inflexion, for $f''(Z)$ changes sign as x passes thru this value.

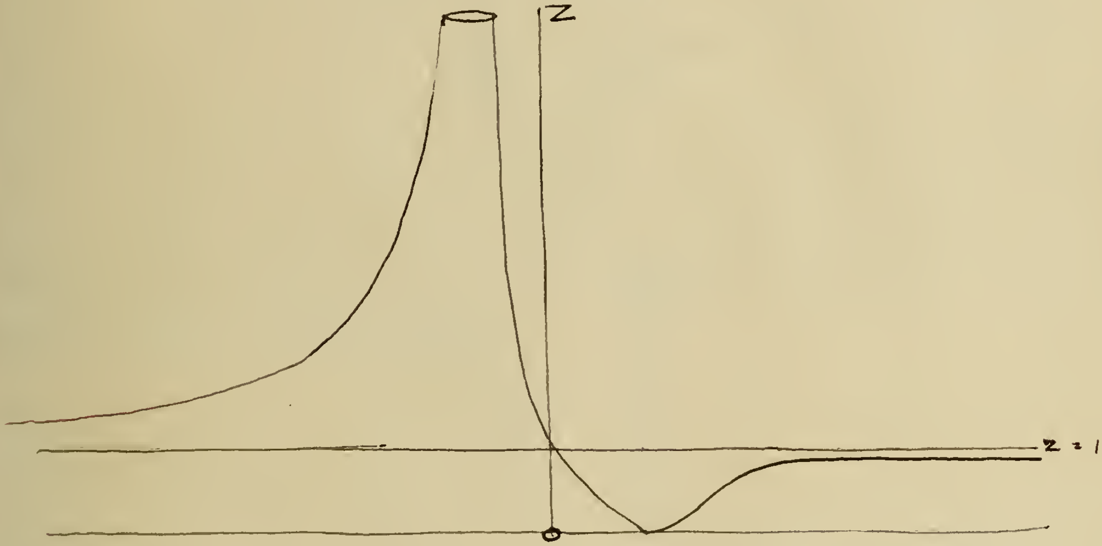
If $Z = 0$, then $(x^2+ y^2-1)^2+ 4y^2= 0$ is the equation ofthe curve of intersection of the surface and the xy plane.

$$(x^2+ y^2- 1)^2 = -4y^2$$

$$x^2+ y^2- 1 = \pm\sqrt{-4y^2} = \pm 2iy.$$

$\therefore y = 0$, $x^2= 1$, $x = \pm 1$, Points only.

Figure XII.



The above figure will give a general idea of the appearance of the surface.

To determine whether or not the xy plane is tangent to the surface at (1,0,0).

$$\frac{\partial f}{\partial z_1}(z-z_1) + \frac{\partial f}{\partial x_1}(x-x_1) + \frac{\partial f}{\partial y_1}(y-y_1) = 0$$ is the equation of the tangent plane to $f(x,y,z) = 0$.

$$f(x,y,z) \equiv Z\left\{(x+1)^2 + y^2\right\} - \left\{(x-1)^2 + y^2\right\} = 0$$

$$\frac{\partial f}{\partial x_1} = 2yz - 2y; \quad \frac{\partial f}{\partial y_1} = 0.$$

$$\frac{\partial f}{\partial z} = \left\{(x+1)^2 + y^2\right\} \quad \frac{\partial f}{\partial z} = 4.$$

Therefore the tangent plane at (1,0,0) is $4Z = 0$, or $Z = 0$, the

xy plane.

To determine the minimum points of the surface. Write the equation in the explicit form,

$$Z = \frac{(x-1)^2 + y^2}{(x+1)^2 + y^2}$$

Taking $\frac{\partial z}{\partial x}$ we get $\frac{2(x-1)\left[(x+1)^2+y^2\right] - \left[(x-1)^2 + y^2\right]2(x+1)}{\left[(x+1)^2 + y^2\right]^2}$

And $\frac{\partial^2 z}{\partial x^2}$

$$= \frac{\left\{2\left[(x+1)^2 + y^2\right] + 2(x-1)2(x+1) - 2(x-1)2(x+1) - \left[(x+1)^2 + y^2\right]2\right\}\left\{\left[(x+1)^2 + y^2\right]^2 - \left\{2(x-1)\left[(x+1)^2 + y^2\right] - \left[(x-1)^2 + y^2\right]2(x+1)\right\}\right.\left.2\left[(x+1)^2 + y^2\right]2(x+1)\right\}}{\left[(x+1)^2 + y^2\right]^2}$$

Computing, similarly, $\frac{\partial z}{\partial y}$ and $\frac{\partial^2 z}{\partial y^2}$ and substituting (1,0,0), the coordinates of the point of tangency, for x,y,z, in these expressions; we see that the first partial derivatives vanish, while the second partial derivatives are positive: therefore (1,0,0) is a minimum point of the surface.

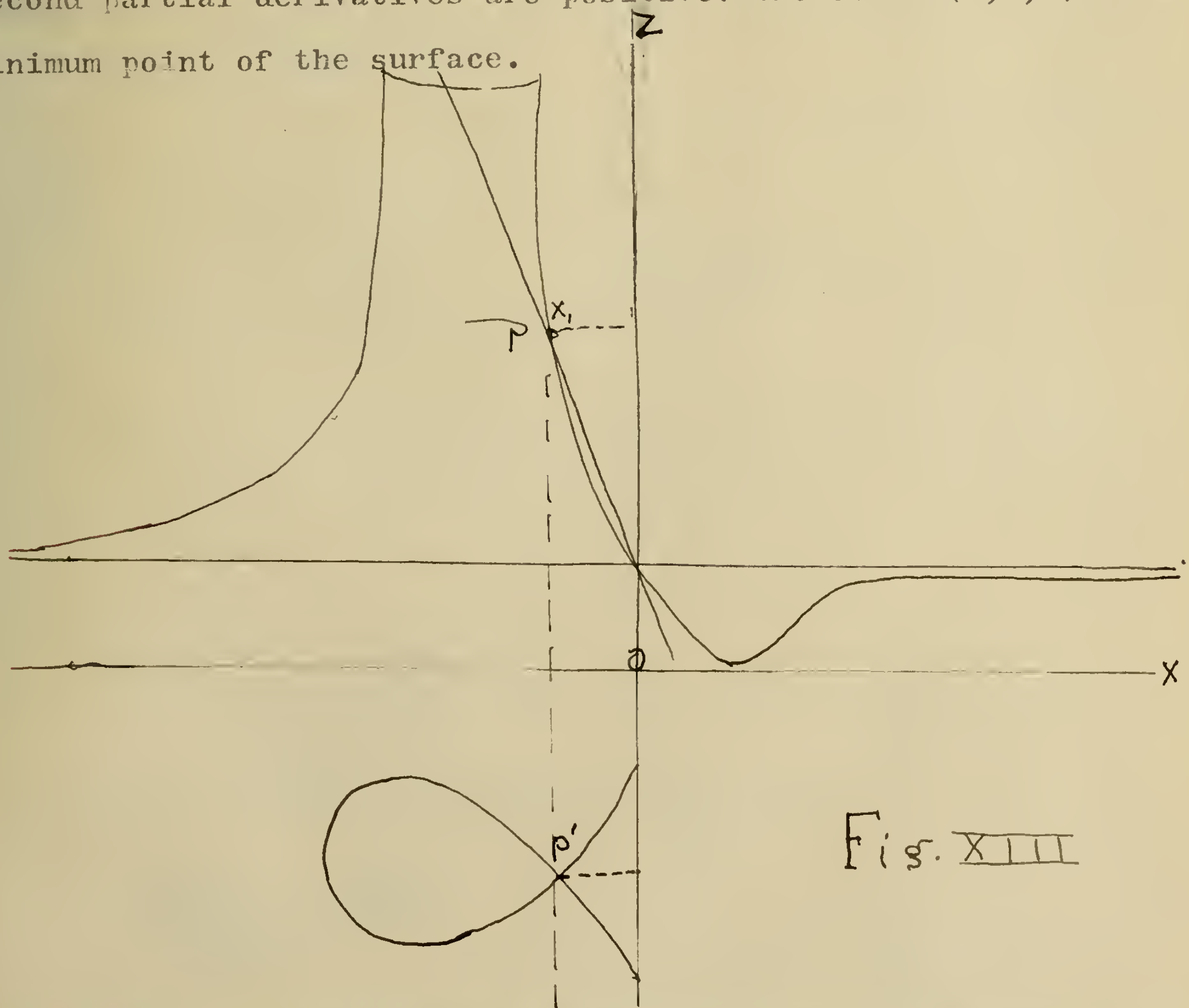


Fig. XIII

This shows the cross section of xz plane.

The equation of the intersecting tangent plane at P is

$$z-z_1 = \frac{dz}{dx_1}(x-x_1)$$

To find the equation of the projection in the xy plane of the curve of intersection, eliminate z between

1) $Z\left[(x+1)^2 + y^2\right] - \left[(x-1)^2 + y^2\right] = 0$ and

2) $Z-Z_1 = \frac{dz}{dx_1}(x-x_1)$

3) $Z = Z_1 - \frac{dz}{dx_1}(x-x_1)$

$$\left[Z_1 + \frac{dz}{dx}(x-x_1)\right]\left[(x+1)^2 + y^2\right] - \left[(x-1)^2 + y^2\right] = 0$$

is the equation of the projection of the curve of intersection of the surface and xy plane.

To find the asymptotes to this curve, put for brevity $Z_1 = k_1$, $\frac{dz}{dx} = k_2$, and $x_1 = k_3$. Then expanding and simplifying the equation of this curve becomes

$$k_1x^3 + (k_1 + 2k_2 - k_2k_3 - 1)x^2 + (2k_1 + k_2 - 2k_2k_3 + 2)x + (k_1 - k_2k_3 - 1)y^2 + k_2xy^2 + (k_1 - k_2k_3 - 1) = 0 \equiv f(x,y).$$

Arranging this equation in descending powers of y, we see that it is a cubic with the y^3 term missing. Thus

$$0.y^3 + (k_1 - k_2k_3 - 1 + k_2x)y^2 + \text{------------} = 0$$

The equation of the asymptote is therefore, $k_1 - k_2k_3 - 1 + k_2x = 0$.

To reduce this we have

$k_2x = 1 + k_2k_3 - k_1$ or

$$x = \frac{1 + k_2k_3 - k_1}{k_2} = \frac{1 + \frac{dz}{dx}x_1 - z_1}{\frac{dz}{dx}}$$

and substituting the expressions for $\frac{dz}{dx}$, and Z :

$$x = \frac{1 + \frac{4(x_1-1)x_1}{(x_1+1)^3} - \frac{(x_1-1)^2}{(x_1+1)^2}}{\frac{4(x_1-1)}{(x_1+1)^3}}$$

$$= \frac{(x_1+1)^3 + 4(x_1-1)-(x_1-1)^2(x+1)}{4(x_1-1)}$$

$$= \frac{x_1^3 + 3x_1^2 + 3x_1 + 1 + 4x_1^2 - 4x_1 - x_1^3 + x_1^2 + x_1 - 1}{4(x_1-1)}$$

$$= \frac{8x_1^2}{4(x_1-1)} \quad = \frac{2x_1^2}{x_1-1}$$

Explicitly the equation of the asymptote is now

$$x = \frac{2x_1^2}{x_1-1}$$

To find the slope of the tangent at the double point and determine whether or not the angle between the tangents is constant.

$$k_2x^3 + (k_1 + 2k_2 - k_2k_3 - 1)x^2 + (2k_1 + k_2 - 2k_2k_3 + 2)x + (k_1 - k_2k_3 - 1)y^2$$
$$+ k_2xy^2 + (k_1 - k_2k_3 - 1) = 0 \equiv f(x,y)$$

the projection in the xy plane of the curve of intersection of the surface and tangent plane. Dividing thru by k_2 the equation becomes:

$$x^3 + \frac{(k_1 + 2k_2 - k_2k_3 - 1)x^2}{k_2} + \frac{(2k_1 + k_2 - 2k_2k_3 + 2)x}{k_2.} + \frac{(k_1 - k_2k_3 - 1)}{k_2}y^2$$
$$+ xy^2 + \frac{(k_1 - k_2k_3 - 1)}{k_2} = 0$$

Or, for brevity we may write this equation in the form

$$x^3 + ax^2 + bx + cy^2 + xy^2 + c = 0$$

$$y^2(c+x) = -x^3 - ax^2 - bx - c$$

$$y^2 = - \frac{x^3 + ax^2 + bx + c}{x+c}$$

Transforming the origin to the double point we have as the equations of transformation

$$\left.\begin{aligned} x &= x_1 + x' \\ y &= y' \end{aligned}\right\}$$

Substituting these values in the previous equation we get

$$y'^2 = - \frac{(x_1 + x')^3 + a(x_1 + x')^2 + b(x_1 + x') + c}{x_1 + x' + c}$$

Dropping primes and dividing both sides of the equation by x^2, we get

$$\frac{y^2}{x^2} = - \frac{(x_1 + x)^3 + a(x_1 + x)^2 + b(x_1 + x) + c}{x^2(x_1 + x + c)} = \left(\frac{y}{x}\right)^2$$

And $\lim\limits_{x \doteq 0}\left(\frac{y}{x}\right)^2 = \lim\limits_{y \doteq 0} \frac{-(x_1 + x)^3 + a(x_1 - x)^2 + b(x_1 + x) + c}{x^2(x_1 + x + c)}$

Now replacing a, b, and c by their values in terms of x, & Z we get for this quotient

$$(x_1 + x)^3 + (x_1^2 + 2x_1 x + x^2)\left(Z_1 + \frac{\frac{2dZ_1}{dx} - \frac{dZ_1}{dx}x_1 - 1}{}\right) + (x_1 + x)\left(2Z_1 - \frac{\frac{2dZ_1}{dx}x + \frac{dZ_1}{dx} + 2}{\frac{dZ_1}{dx}}\right) + Z_1 - \frac{\frac{DZ_1}{dx}x_1 - 1}{\frac{dZ_1}{dx}}$$

, all over the denominator

$$x^2\left(x + x_1 + \frac{\left(Z_1 - \frac{dZ_1}{dx}x_1 - 1\right)}{\frac{dZ_1}{dx}}\right)$$

Reducing this by terms,

$$\frac{Z_1 - \frac{2dZ_1}{dx} - \frac{dZ_1}{dx}x_1 1}{\frac{dZ_1}{dx}} = \frac{\frac{(x_1 - 1)^2}{(x_1+1)^2} + \frac{8(x_1 - 1)}{(x_1+1)^3} - 4\frac{(x_1 - 1)x_1 - 1}{(x_1+1)^3}}{\frac{4(x_1 - 1)}{(x_1+1)^3}}$$

$$= \frac{(x_1 - 1)^2(x_1 + 1) + 8(x_1 - 1) - 4)x_1 - 1)x_1 - (x_1 + 1)^3}{4(x_1 - 1)}$$

$$= \frac{(x_1^2 - 2x_1 + 1)(x_1 + 1) + 8x_1 - 8 - 4x_1^2 + 4x_1 - x_1^3 - 3x_1^2 - 1}{4(x_1 - 1)}$$

$$= \frac{-8x_1^2 + 8x_1 - 8}{} = \frac{2x_1^2 + 2x_1 - 2}{x_1 - 1}$$

And,

$$\frac{2Z_1 - 2\frac{dZ_1}{dx}x_1 + \frac{dZ_1}{dx} + 2}{\frac{dZ_1}{dx}} = \frac{\frac{2)x_1 - 1)^2}{(x_1+1)^2} - \frac{8x_1(x_1 - 1)}{(x_1+1)^3} + \frac{4(x_1 - 1)}{(x_1+1)^3} + 2}{\frac{4(x_1 - 1)}{(x_1+1)^3}}$$

$$= \frac{2(x_1 - 1)^2(x_1 + 1) - 8x_1(x_1 - 1) + 4(x_1 - 1) + 2)x_1 + 1)^3}{4(x_1 - 1)}$$

$$= \frac{2(x_1^2 - 2x_1 + 1)(x_1 + 1) - 8x_1^2 + 8x_1 + 4x_1 - 4 + 2x_1^3 + 6x_1^2 + 6x_1 + 2}{4x_1 - 4}$$

$$= \frac{4x_1^3 - 4x_1^2 + 16x_1}{4x_1 - 4}$$

$$= \frac{x_1^3 - x_1^2 + 4x_1}{x_1 - 1}$$

And,

$$\frac{Z_1 - \frac{dZ_1}{dx} x_1 - 1}{\frac{dZ_1}{dx}} = \frac{\frac{(x_1 - 1)^2}{(x_1 + 1)^2} h \frac{4x_1 (x_1 - 1)}{(x_1 + 1)^3} - 1 \quad Y}{\frac{4(x_1 - 1)}{(x_1 + 1)^3}}$$

$$= \frac{(x_1 - 1)^2 (x_1 + 1) - 4x_1 (x_1 - 1) - (x_1 + 1)^3}{4(x_1 - 1)}$$

$$= \frac{(x_1^2 - 2x_1 + 1)(x_1 + 1) - 4x_1^2 + 4x_1^2 - x_1^3 - 3x_1^2 - 3x_1 - 1}{4(x_1 - 1)}$$

$$= \frac{-8x_1^2}{4(x_1 - 1)} = \frac{-2x_1^2}{x_1 - 1}$$

Substituting these values of the terms in reduced form in the equation, we get

$$\frac{(x_1 + x)^3 +)x_1^2 + 2x_1 x + x^2) \frac{(-2x_1^2 + 2x_1 - 2)}{x_1 - 1} + (x_1 + x)}{\frac{\frac{(x_1^3 - x_1^2 + 4x_1)}{x_1 - 1} - \frac{2x^2}{x_1 - 1}}{x2(2x + x - \frac{2x_1)}{x_1 - 1}}}$$

$$= \frac{(x_1^3 + 3x_1^2 x + 3x_1 x^2 + x^3)(x_1 - 1) + 2x_1^3 - 2x_1^4 - 2x_1^2 - 4x_1^3 x - 4x_1^2 x - 4x_1 x - 2x_1^2 x^2}{+ 2x_1 x^2 - 2x^2 + x_1^4 - x_1^3 + 4x_1^2 + x_1^2 x - x_1^2 x + 4x_1 x - 2x_1^2}$$

$$= \frac{x_1^2 x^2 + x_1 x^2 - x_1 x^2 - x^3 - 2x^2}{x_1^2 x^2 + x_1 x^3 - x_1 x^2 - x^3 - 2x_1^2 x^2} = \frac{x_1^2 + x_1 x - x_1 - x - 2}{x_1^2 - x_1 x - x_1 - x - 2x_1^2}$$

$$\lim_{x = o} \frac{(y)^2}{x} = \lim_{x = o} \frac{-x_1^2 + x_1 x - x_1 - x - 2}{x_1^2 + x_1 x - x_1 - x - 2x^2} = \frac{x_1^2 - x_1 - 2}{x_1^2 + x_1} = \frac{x_1 - 2}{x_1}$$

But the slope of the tangents at this point is $\lim_{x \to 0}\left(\frac{y}{x}\right)$, therefore $\sqrt{\frac{x_1 - 2}{x_1}}$ is slope, and this is finite but not constnat.

We have seent that the system of all circles on the surface projected into a pencil of circles with the imaginary base points (1,i), (-1,i). The lines of fastest descent on the surface are a system of lines orthogonal to the system of circles made by planes parallel to the xy plane. These project into a set of curves orthogonal to the projection of these circles in the xy plane as will be seen from the theorem in Solid Geometry:

" If two straight lines in space, intersect at right angles, and one of these lines is parallel to a given plane, then the projection of these lines in that plane, also intersect at right angles.

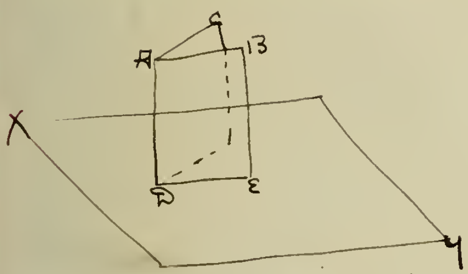


Figure XV.

Thus in Figure XV. we have given the plane xy, A B and AC, straight lines without xy, and AB parallel to XY. Angle CAB = 90,° and DE and DF the projections of AB and AC in xy. According to the theorem, angle FDE is a right angle. Then since the lines of fastest descent on the surface are a system orthogonal to the system of circles formed by the intersection of the surface and planes parallel to the xy plane, the projections of these circles and the projections of lines of fastest descent are orthogonal to each other.

$\left[x + \left(\frac{z+1}{z-1}\right)\right]^2 + y^2 = \frac{4z}{(z-1)^2}$ is the equation of the system of circles. The system of curves orthogonal to this system passes thry the points (1,0) and (-1,0) therefore intersects the x axis. Now we know that if we have two curves f(x) = 0 and g(x) = 0 then every curve thru their intersection is given by $f(x) + \lambda g(x) = 0$. Take the particular circle of the first system, then, that passes thru (1,0) and (-1,0).

Its equation is

$x^2 + y^2 - 1 = 0 = f(x)$; and the x axis is $y = 0 = g(x)$. Then the equation of the system of curves thru their points of intersection is

$$(x^2 + y^2 - 1) - \lambda y = 0; \text{ or}$$

$$x^2 + \left(y - \frac{\lambda}{2}\right)^2 = \frac{\lambda^2}{4} + 1,$$

where λ may have any constant value.

This gives the equation of the projection of the lines of fastest descent on xy plane. To get the equation of their projection in yZ plane solve this equation simultaneously with the equation of the surface, eliminating x.

Between

1) $x^2 + \left(y - \frac{\lambda}{2}\right)^2 = \frac{\lambda^2}{4} + 1$ and

2) $Z\left[(x+1)^2 + y^2\right] - \left[(x-1)^2 + y^2\right] = 0.$

From 1) $x = \pm\sqrt{\frac{\lambda^2}{4} + 1 - \left(y - \frac{\lambda}{2}\right)^2} = \sqrt{1 - y^2 + \lambda y}$

Substituting this value in 2) we get

$$Z\left[\left(\sqrt{1+\lambda y - y^2} + 1\right)^2 + y^2\right] - \left[\left(\sqrt{1 - y^2 + \lambda y} - 1\right)^2 + y^2\right] = 0 \text{ Or,}$$

$$Z\left[(1 + \lambda y) + 1 + 2\sqrt{1+\lambda y - y^2}\right] - \left[1 - y^2 + \lambda y + 1 - 2\sqrt{1 + \lambda y - y^2} + y^2\right] = 0$$

$$Z\left[2 + 2\sqrt{1+\lambda y - y^2} + \lambda y\right] - \left[2 - 2\sqrt{1 - \lambda y + y^2} + \lambda y\right] = 0$$

Expanding: $2Z + 2Z\sqrt{1+\lambda y - y^2} + \lambda y Z - 2 + 2\sqrt{1+\lambda y - y^2} - \lambda y = 0$

$$2Z + Z\lambda y - 2 - \lambda y = -2Z\sqrt{1+\lambda y - y^2} - 2\sqrt{1+\lambda y - y^2}$$

Or, collecting terms:

$$2(Z-1) + \lambda y(Z-1) = -(2Z+2)\sqrt{1+\lambda y - y^2}$$

$$\frac{(2+\lambda y)(Z-1)}{2Z+2} = -\sqrt{1+\lambda y - y^2} \qquad \text{Squaring:}$$

$$\frac{(2+\lambda y)^2 (Z-1)^2}{(2Z+2)^2} = 1 + \lambda y - y^2 \text{ Or,}$$

$$(4 + 4\lambda y + \lambda^2 y^2)(Z^2 - 2Z + 1) = (1 + \lambda y - y^2)4(Z^2 + 2Z + 1)$$

Or $4Z^2 + 4\lambda y Z^2 + Z^2 y^2 \lambda^2 - 8Z - 8\lambda y Z - 2\lambda^2 y^2 Z + 4 + 4\lambda y + \lambda^2 y^2 =$

$4Z^2+8Z+4+4\lambda yZ^2+8\lambda yZ+4\lambda y-4y^2Z^2-8y^2Z-4y^2$

Collecting terms: we get for the required equation

(6) $(\lambda^2+4)y^2Z^2-2(\lambda^2-4)y^2Z-16\lambda yZ+(\lambda^2+4)y^2-16Z = 0$

The projection of a curve of fastest descent upon the yZ plane is therefore a quartic.

That the curve must generally be a quartic is also geometrically apparent. As both surfaces pass thru the absolute at infinity, their curve of intersection being of the sixth order, degenerates into a quartic and the infinite imaginary circle, absolute.

To determine the asymptotes to this curve, put $y = mZ + b$.

Then

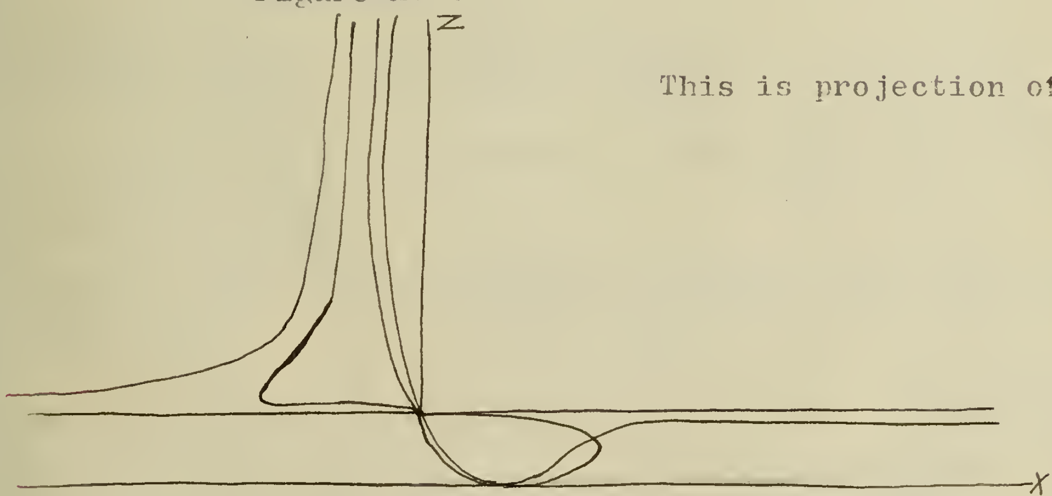
$(\lambda^2+4)Z^2(mZ+b)^2-2(\lambda^2-4)Z(mZ+b)^2-16\lambda Z(mZ+b)+(\lambda^2+4)(mZ+b)^2-16Z = 0.$

Expanding and collecting terms

$(\lambda^2m^2+4m^2)Z^4+(2\lambda^2mb+8mb-2\lambda^2m^2+8m^2)Z^3+ \text{---------} = 0.$

$\lambda^2m^2+4m^2=0$, $m^2=0$. ; $m=0$ $\therefore b=0$. $\therefore y=0$ Asymptote.

Figure XV .



This is projection on xZ plane.

The above figure illustrates the circles and line of fastest descent. To show that the curve of intersection of the surface and yZ plane is tangent to xy plane at (x=1,y=0). We have previously shown that the surface is tangent at this point. Therefore if the curve passes thru this point it must be a point of tangency. The coordinates (1,0) satisfy the equation of the curve, therefore (1,0) is a point of tangency.

The tangent plane at (0,0,1) is

$4x+Z-1=0$ for $\frac{\partial f}{\partial x} = 2(x+1)Z-2(x-1)$ for $x = 0, Z= 1$ this becomes

$2Z+2=4$

$\frac{\partial f}{\partial y} = 2Zy-2y = 0$, for $y= 0$, $Z= 1$.

$\frac{\partial f}{\partial Z} = \left[(x+1)^2 + y^2\right]$, $x = 0$, $y= 0$, becomes 1.

$\therefore 4x+Z-1 = 0$ is the tangent plane at (0,0,1).

To getthe equation of the projection of the lines of fastest descent in the xZ plane eliminate y between

1) $Z\left[(x+1)^2 + y^2\right] - \left[(x-1)^2 + y^2\right] = 0$ and

2) $x^2 + \left(y - \frac{\lambda}{2}\right)^2 = \frac{\lambda^2}{4} + 1$

From 1) $Y^2 = \frac{-Z(x+1)^2 + (x-1)^2}{Z-1}$

$\therefore y = \pm\sqrt{\frac{-Z(x+1)^2 + (x-1)^2}{Z-1}}$

And substituting this value in 2) we get

$$x^2 + \frac{-Z(x+1)^2 + (x-1)^2}{Z-1} - \lambda\sqrt{\frac{-Z(x+1)^2 + (x-1)^2}{Z-1}} = 1.$$

Or, expanding, removing parentheses, collecting terms, transposing and combining, we get:

$$(7)\quad (2-\lambda^2)x^2Z^2 + (4+2\lambda^2)xZ^2 + (4-2\lambda^2)x^2Z + (2+\lambda^2)x^2 - (4+2\lambda^2)x + (2+\lambda^2)Z^2$$
$$-(4+2\lambda^2)Z + 2 = 0.$$

This curve is evidently also a quartic. For some definite value of λ let K_λ be the horizontal projection of the quartic, which is of cours a circle of the conjugate pencil, Figure , XVII.

It is apparent that the curve has a double point at $x = 0$, $Z=1$.

For the slopes of the tangents at this point D we get

$$\frac{dz}{dx} = -\frac{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial x \partial z}\cdot\frac{dz}{dx}}{\frac{\partial^2 f}{\partial x \partial z} + \frac{\partial^2 f}{\partial z^2}\cdot\frac{dz}{dx}} \qquad \text{Or,}$$

$$\frac{\partial^2 f}{\partial x\,\partial z}\cdot\frac{dz}{dx}+\frac{\partial^2 f}{\partial z^2}\left(\frac{dz}{dx}\right)^2+\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial x\,\partial z}\cdot\frac{dz}{dx}=0.$$

$$\text{Or } \frac{\partial^2 f}{\partial z^2}\left(\frac{dz}{dx}\right)^2+\frac{2\partial^2 f}{\partial x\,\partial z}\cdot\frac{dz}{dx}+\frac{\partial^2 f}{\partial x^2}=0.$$

$$1)\ \therefore \frac{dz}{dx}=\frac{\dfrac{-2\partial^2 f}{\partial x\,\partial z}\pm\sqrt{4\left(\dfrac{\partial^2 f}{\partial x\,\partial z}\right)^2-\dfrac{4\partial^2 f}{\partial z^2}\cdot\dfrac{\partial^2 f}{\partial x^2}}}{\dfrac{2\partial^2 f}{\partial z^2}}$$

$$2)\ f(x,Z)=(2+\lambda^2)x^2z^2+(4+2\lambda^2)x\,z^2+(4-2\lambda^2)x^2z+(2+\lambda^2)x^2-(4+2\lambda^2)x+$$

$$(2+\lambda^2)z^2-(4+2\lambda^2)z+(2+\lambda^2)(\ \ -$$

$$\frac{\partial f}{\partial z}=2(2+\lambda^2)x^2z+2(4+2\lambda^2)xz+(4-2\lambda^2)x^2+2(2+\lambda^2)Z-(4+2\lambda^2)$$

$$\frac{\partial f}{\partial x}=2(2+\lambda^2)xZ^2+(4+2\lambda^2)Z^2+2(4-2\lambda^2)xZ+2(2+\lambda^2)x-(4+2\lambda^2)$$

$$\frac{\partial^2 f}{\partial x^2}=2(2+\lambda^2)Z^2+2(4-2\lambda^2)Z+2(2+\lambda^2)$$

$$\frac{\partial^2 f}{\partial z^2}=2(2+\lambda^2)x^2+2(4+2\lambda^2)Z+2(2+\lambda^2)$$

$$\frac{\partial^2 f}{\partial x\,\partial Z}=4(2+\lambda^2)xZ+2(4+2\lambda^2)Z+2(4-2\lambda^2)x\,,$$

Substituting these values along with $x=0$, $z=1$, in 1). We get

$$\frac{dz}{dx}=\frac{-2(2+\lambda^2)\pm 2\lambda\sqrt{\lambda^2-2}}{2+\lambda^2}$$

$$\therefore m_1=\frac{-2(2+\lambda^2)+2\lambda\sqrt{\lambda^2-2}}{2+\lambda^2},\ \text{and}$$

$$m_2=\frac{-2(2+\lambda^2)-2\lambda\sqrt{\lambda^2-2}}{2+\lambda^2}$$

For $\lambda=0$ these slopes have the same value, viz. -2. Then K_λ is symmetrical with respect to the xZ plane, and its projection on the xZ plane consequently degenerates into an hyperbola, as we see by replacing λ by 0 in 2). This gives

$$2x^2z^2+4xZ^2+4x^2Z+2x^2-4x+2Z^2-4Zx^2=0$$

$$\text{Or, } x^2z^2+2xz^2+2x^2z+x^2-2x+z^2-2z+1=0$$

$$(xz+x+z-1)^2=0$$

$$(xz+x+z-1)=0$$

$$Z(x+1)+(x-1)=0,$$

(9) $Z = -\frac{x-1}{x+1} = \frac{1-x}{1+x}$

for the equation of the hyperbola. For λ the intersection of the surface and the xz plane is obtained. The quartic degenerates into the cubic

(10) $Z(x+1)^2 - (x-1)^2 = 0$ and the line $Z = 1$.

Figure XVII

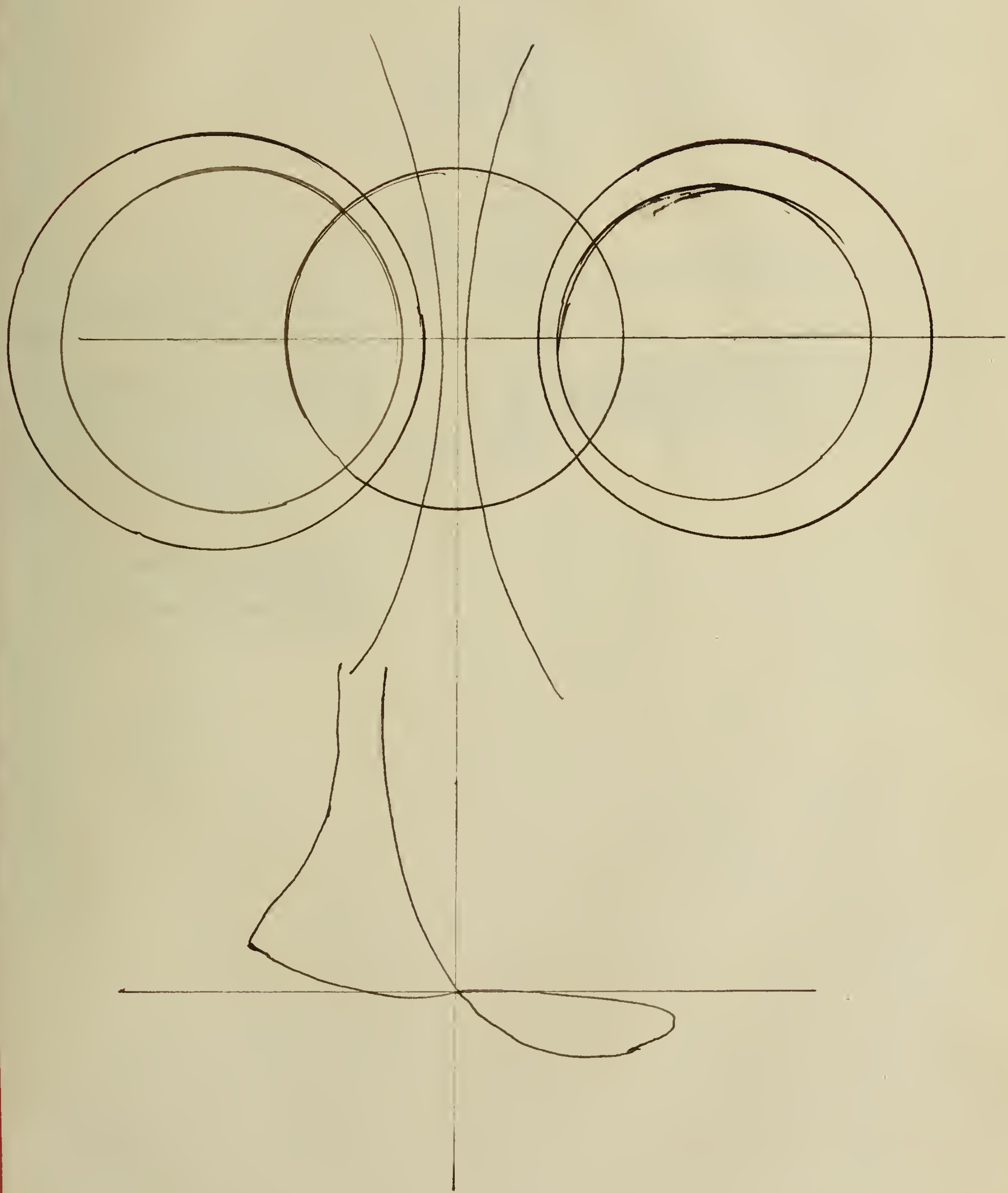
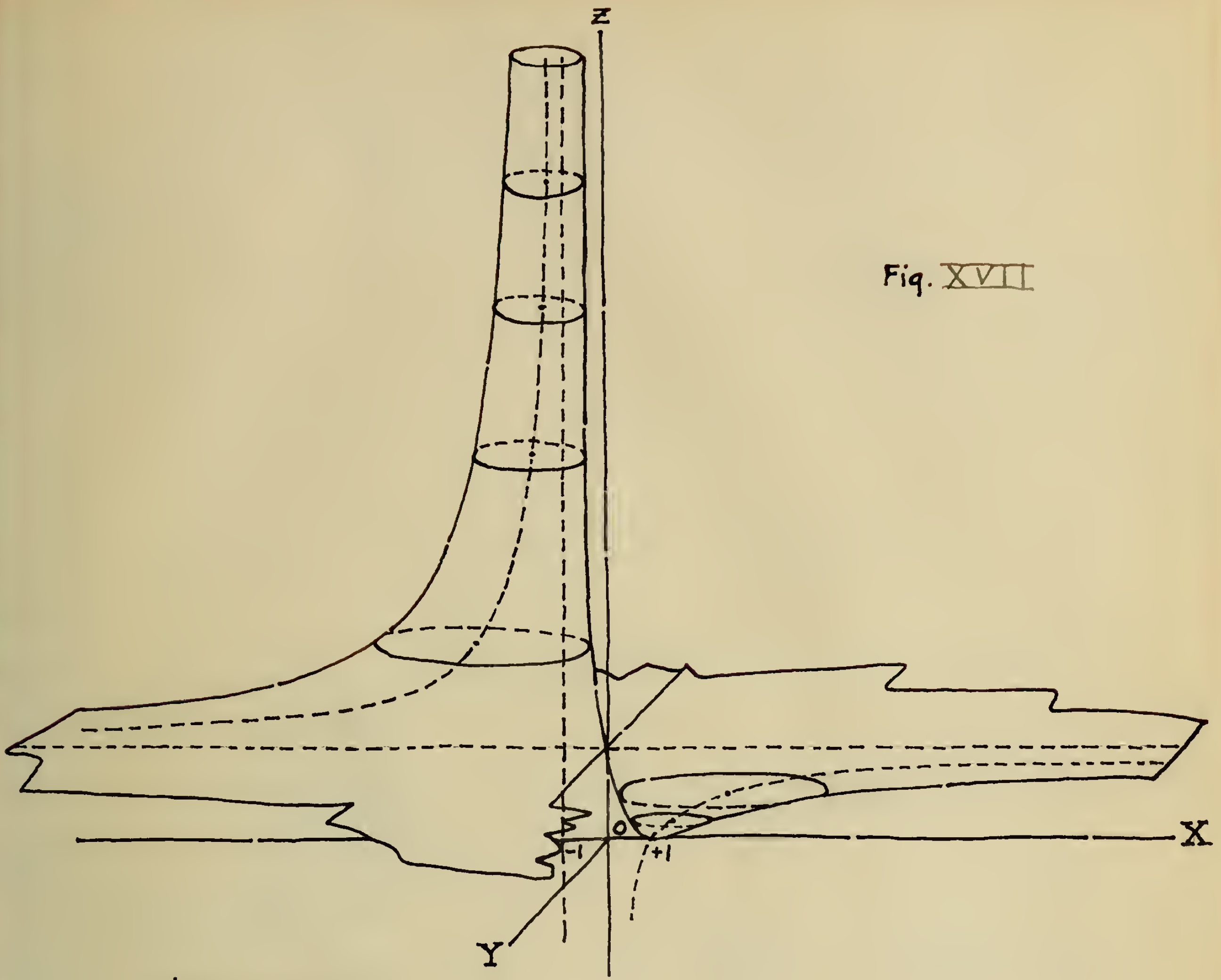


Fig. XVII



unit on Z axis
= 2 units on X axis

IV. (1) $W = z^3 - 1$

$u+iv = (x+iy)^3 - 1$

$= x^3 + 3ix^2y - 3xy^2 - iy^3 - 1$

$u = x^3 - 3xy^2 - 1$

$v = 3x^2y - y^3$

$u^2 + v^2 = (x^3 - 3xy^2 - 1)^2 + (3x^2y - y^3)^2$

$= x^6 + 9x^2y^4 + 1 - 6x^4y^2 - 2x^3 + 6xy^2 + 9x^4y^2 + y^6 - 6x^2y^4$

$\therefore Z = x^6 + 3x^4y^2 - 2x^3 + 3x^2y^4 + 6xy^2 + 1 + y^6$

is the equation of the surface.

As for $W = -1$, three roots of the equation in Z,

$z^3 - 1 - W = 0$

become equal to zero, then to the unit circle $|W|^2 = 1$ thru -1 in the W-plane corresponds in the xy-plane a curve which, according to the general theory has a triple-point at the origin and three branches at equal angles of 60°.

To find this curve of intersection of the surface and a plane parallel to the xy plane at a distance 1. Solve simultaneously 1) $z = 1$ and 2)

$Z = x^6 + 3x^4y^2 - 2x^3 + 3x^2y^4 + 6xy^2 + 1 + y^6,$

which gives:

3) $f(x,y) = x^6 + 3x^4y^2 - 2x^3 + 3x^2y^4 + 6xy^2 + y^6 = 0.$

This has no real asymptotes, as we see by substituting $y = mx+b$, and solving.

For $\lim\limits_{x \doteq \infty} \left\{\frac{f(x,y)}{x^6}\right\}$ we get

$\lim\limits_{x \doteq \infty} \left\{1 + 3\left(\frac{y}{x}\right)^2 + 3\left(\frac{y}{x}\right)^4 + \left(\frac{y}{x}\right)^6\right\},$

or writing $\lim\limits_{\substack{x \doteq \infty \\ y \doteq \infty}} \left(\frac{y}{x}\right) = m$

$(1+m^2)^3$. Consequently, $(m = \pm i,\ x \doteq \infty,\ y \doteq \infty)$ satisfy the equation. The circular points are triple points of the sextic. This curve is

therefore tricircular.

Since there is, in the equation of this curve, no constant term, it passes through the origin. Reducing the equation of the curve of intersection of the surface and the plane $Z = 1$, to polar coordinates we get:

$$\rho^6(\cos\theta)^6 + 3(\rho\cos\theta)^4(\rho\sin\theta)^2 - 2(\rho\cos\theta)^3 + 3(\rho\cos\theta)^2(\rho\sin\theta)^4$$
$$+ 6(\rho\cos\theta)(\rho\sin\theta)^2 + (\rho\sin\theta)^6 = 0 \quad \text{expanding,} =$$
$$\rho^6\cos^6\theta + 3\rho^6\cos^4\theta\sin^2\theta - 2\rho^3\cos^3\theta + 3\rho^6\cos^2\theta\sin^4\theta + 6\rho^3\cos^6\theta\sin^2\theta + \rho^6\sin^6\theta = 0$$
$$= F(\rho, \theta).$$
$$= \rho^6(\sin^2\theta + \cos^2\theta)^3 + \rho^3 2\cos\theta\,(3 - 4\cos^2\theta) = 0$$

Finally, (4) $\rho^6 + \rho^3\, 2\cos\theta\,(3 - 4\cos^2\theta) = 0$

To plot this, we calculate the following table of values:

θ	ρ
0	1.26
$\frac{\pi}{12}$	1.12
$\frac{\pi}{6}$	0.
$\frac{3\pi}{12}$	-1.12
$\frac{\pi}{3}$	-1.26
$\frac{5\pi}{12}$	±1.12
$\frac{\pi}{2}$	0.

The surface has the property of axial symmetry with respect to the Z axis with respect to a rotation thru 120^0. This is determined by subjecting the equation of the surface to the transformations

$$x = x'\cos\tfrac{2\pi}{3} - y'\sin\tfrac{2\pi}{3}$$
$$y = x'\sin\tfrac{2\pi}{3} + y'\cos\tfrac{2\pi}{3}$$

Substituting these values in

$$Z = x^6 + 3x^4y^2 - 2x^3 + 3x^2y^4 + 6xy^2 + 1 + y^6$$

we get

$$\left(\frac{-x'}{2} - y'\frac{\sqrt{3}}{2}\right)^6 + 3\left(\frac{-x'}{2} - y'\frac{\sqrt{3}}{2}\right)^4\left(\frac{x'\sqrt{3}}{2} - \frac{y'}{2}\right)^2 - 2\left(\frac{-x'}{2} - \frac{y'\sqrt{3}}{2}\right)^3 + 3\left(\frac{-x'}{2} - y'\frac{\sqrt{3}}{2}\right)^2$$

$$\left(\frac{x'\sqrt{3}}{2} - \frac{y'}{2}\right)^4 + 6\left(\frac{-x'}{2} - y'\frac{\sqrt{3}}{2}\right)\left(\frac{x'\sqrt{3}}{2} - \frac{y'}{2}\right)^2 + 1 + \left(\frac{x'\sqrt{3}}{2} - \frac{y'}{2}\right)^6 = Z$$

Expanding, collecting terms and simplifying we get

$$Z = \frac{1}{64}(x^6 + 6x^5y\sqrt{3} + 45x^4y^2 + 60\sqrt{3}x^3y^3 + 135x^2y^4 + 54\sqrt{3}xy^5 + 27y^6) + \frac{3}{64}(3x^6$$
$$+ 10x^5y\sqrt{3} + 31x^4y^2 + 4x^3y^3\sqrt{3} - 27x^2y^4 - 6xy^5\sqrt{3} + 9y^6) + \frac{1}{4}(x^3 + 3x^2y\sqrt{3} + 9xy^2$$
$$+ 3y^3\sqrt{3}) + \frac{3}{64}(9x^6 + 6x^5y\sqrt{3} - 27x^4y^2 - 4x^3y^3\sqrt{3} + 31x^2y^4 - 10xy^5\sqrt{3} + 3y^6) + \frac{3}{4}(-3x^3 - x^2y\sqrt{3}$$
$$+ 5xy^2 - y^3\sqrt{3}) + 1 + \frac{1}{64}(27x^6 - 54x^5y\sqrt{3} + 135x^4y^2 - 60x^3y^3\sqrt{3} + 45x^2y^4 - 6xy^5\sqrt{3} + y^6)$$
$$= x^6 + 3x^4y^2 + 3x^2y^4 + y^6 + 1 - 2x^3 + 6xy^2 = Z$$

Therefore a rotation thru 120° about the Z-axis leaves the surface invariant. Since this is true, curves of the cross sections may be plotted thru 120° and then repeated.

To find the intersection of the surface and the xZ-plane, put $y = 0$ in the equation of the surface and we get

(5) $x^6 - 2x^3 + 1 = Z$.

To plot this we calculate the following table of values:

x	Z
0	1
1	0
-1	4
-2	81
1/2	$\frac{49}{64}$
$\frac{3}{2}$	16-
$\frac{1}{8}$	1.003
$\frac{1}{4}$	.96
$\frac{3}{8}$	.89

x	Z
$\frac{5}{8}$	.56
$\frac{3}{4}$	.33
$\frac{7}{8}$	.109
2	49
$-\frac{1}{2}$	1.27

Similarly for the intersection of the surface and yZ plane, put $x = 0$ and we get

(6) $y^6 + 1 = Z$

and have the following table of values.

y	Z
0	1
1	2
$\frac{1}{2}$	$\frac{11}{64}$
$\frac{3}{2}$	11
2	63
-1	2
$-\frac{1}{2}$	$\frac{11}{64}$

$z = (x^3 - 1)^2$

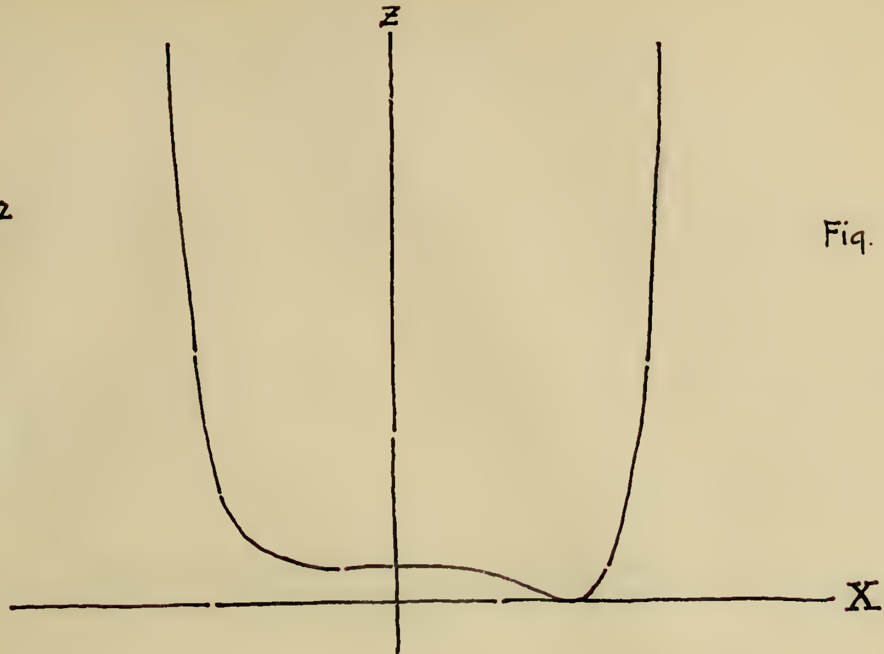


Fig. XVIII

$z = y^6 + 1$

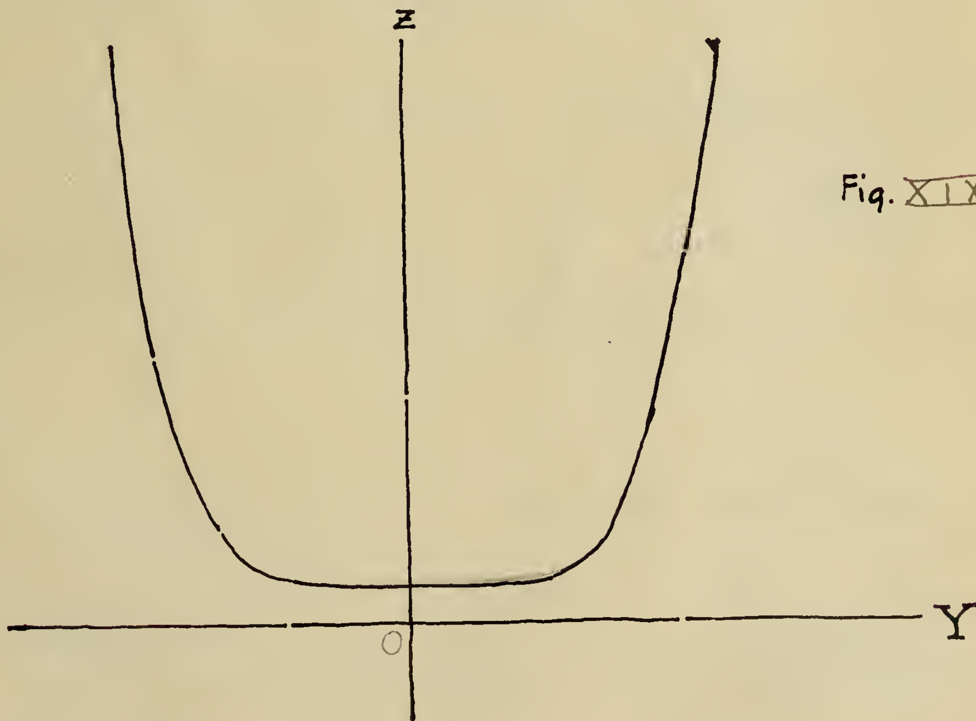


Fig. XIX

scale on X and Y axes
5 times scale on Z axis

To find the points of inflexion of $x^6 - 2x^3 + 1 = Z$, we have the condition that the second derivative must vanish.

$f'(x) = 6x^5 - 6x^2$

$f''(x) = 30x^4 - 12x$

which vanishes for $x = 0$. $\therefore (0,1)$ is a point of inflexion of the curve.

To find the tangent at this point substitute in the general equation of the tangent

$(z-z_1) = \frac{dz}{dx_1}(x-x_1)$ the values $(0,1)$ for (x, Z_1) and we get $(z-1) = 0$,

Therefore in its explicit form $Z = 1$ is the equation of the tangent to the curve at the point of inflexion in the xZ plane.

To find the points of inflexion of $Z = y^6 + 1$.

Using the same method as before we obtain

$f'(y) = 6y^5$

$f''(y) = 30y^4$, which vanishes for $y = 0$, therefore $(0,1)$ is a point of inflexion.

To find the tangent to the curve at this point, put $(0,1)$ for y, Z in the general equation of the tangent, $(Z-Z_1) = \frac{dZ}{dy}(y-y_1)$ and we get:

$Z-1=0$.

Therefore $Z = 1$ in the yZ plane is the equation of the tangent to

$y^6 + 1 = Z$,

at the point of inflexion.

To prove that the surface is tangent to the xy plane at $(1,0)$, $(-\frac{1}{2}, \frac{1}{2}\sqrt{3})$ $(-\frac{1}{2}, -\frac{1}{2}\sqrt{3})$.

$Z = x^6 + 3x^4y^2 - 2x^3 + 3x^2y^4 + 6xy^2 + 1 + y^6$ surface.

$Z-Z_1 = \frac{\partial z}{\partial x_1}(x - x_1) + \frac{\partial Z}{\partial y}(y - y_1)$ equation of the tangent plane at

(x, y, Z_1).

$\frac{\partial Z}{\partial x} = 6x^5 + 12x^3y^2 - 6x^2 + 6xy^4 + 6y^2$; $\frac{\partial Z}{\partial x_1} = 0$

$\frac{\partial Z}{\partial y} = 6x^4y + 12x^2y^3 + 12xy + 6y^5$; $\frac{\partial Z}{\partial y_1} = 0$

$Z - 0 = 0 \therefore Z = 0$ is the equation of tangent plane at (1,0).

Similarly substituting for x, & y in turn $(-\frac{1}{2}, \frac{1\sqrt{3}}{2})$ and $(-\frac{1}{2}, -\frac{1\sqrt{3}}{2})$ in the partial derivatives they banish thus showing that the surface is tangent to the xy plane at the three points.

To show that these are minimum points of the surface take the second partial derivatives and we get $\frac{\partial^2 Z}{\partial x^2} = 30x^4 + 36x^2y^2 - 12x + 6y^4$

And $\frac{\partial^2 Z}{\partial y^2} = 6x^4 + 36x^2y^2 + 12x + 30y^4$.

Substituting (1,0) in partial derivative with respect to x we get $\frac{\partial^2 Z}{\partial x^2} = 18$.

Substituting in $\frac{\partial^2 Z}{\partial y^2}$ we get $\frac{\partial^2 Z}{\partial y^2} = 18$. Similarly substituting the values $(-\frac{1}{2}, \frac{1\sqrt{3}}{2})$ and $(-\frac{1}{2}, -\frac{1\sqrt{3}}{2})$ in the partial derivatives we find that in each case their values are positive. Therefore these are minimum points of the surface, since these values in the first partial derived functions reduce them to zero.

To determine the angles at which the curve of the intersection of the surface and the plane Z = 1, curs the xZ plane we form the limit$\left(\frac{y}{x}\right)$, as x and $y \doteq 0$.

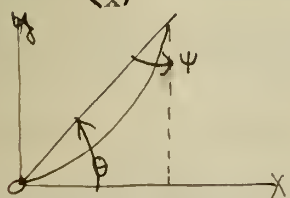


Fig XX

In polar coördinates there is $\cot\psi = \frac{1}{\rho}\frac{d\rho}{d\theta}$ and from

$\rho^3 = 8\cos^3\theta - 3\cos\theta$

$3\rho^2\frac{d\rho}{d\theta} = 24\cos^2\theta\sin\theta + 6\sin\theta$, or

$\rho^2\frac{d\rho}{d\theta} = 8\cos^2\theta\sin\theta + 2\sin\theta$, or

$\cot\psi = \frac{-8\cos^2\theta\sin\theta + 2\sin\theta}{8\cos\quad - 6\cos}$, from which

$\lim_{\theta=\frac{\pi}{6}}(\cot\psi) = \infty$ and

$\lim_{\theta=\frac{\pi}{2}}(\cot\psi) = \infty$

This shows that the angle between the tangent and the radius vector for $\theta = \frac{\pi}{2}$ or $\frac{\pi}{6}$, is zero. And therefore $\theta = \frac{\pi}{2}$, $\theta \pm \frac{\pi}{6}$ are tangents to the curve as the graph shows.

Consider again this equation

$x^6 + 3x^4y^2 - 2x^3 + 3x^2y^4 + 6xy^2 + y^6 = 0.$

Divide through by x^6, thus getting

$1 + 3\left(\frac{y}{x}\right)^2 - \frac{2}{x^3} + 3\left(\frac{y}{x}\right)^4 + \frac{6y^2}{x^2x^3} + \left(\frac{y}{x}\right)^6 = 0.$

The limit of this as x and $y = \infty$ and $\mu = \lim\left(\frac{y}{x}\right)$:

$1 + 3\left(\frac{y}{x}\right)^2 + 3\left(\frac{y}{x}\right)^4 + \left(\frac{y}{x}\right)^6 = 0$ becomes $(\mu^2 + 1)^3 = 0.$

From this $\mu = \pm i$ and $\mu = i$, $\mu_2 = -i$ are triple roots of the equation.

Therefore the circular points are triple points of the sextic.

Consider the intersection of the surface and the plane Z = 2. Substituting this value in the equation of the surface we get

$x^6 + 3x^4y^2 - 2x^3 + 3x^2y^4 + 6xy^2 + y^6 - 1 = 0$

Writing in polar coordinates, collecting terms and simplifying we have $\rho^6 - \rho^3(6\cos\theta - 8\cos^3\theta) - 1 = 0$

Or $\rho^3 = \frac{-(6\cos\theta - 8\cos^3\theta) \pm \sqrt{(6\cos\theta - 8\cos^3\theta)^2 + 4}}{2}$

Calculating values we get the following table.

[illegible]

ρ	θ	$\cos\theta$
1.37	0	1
1.34	$\frac{\pi}{12}$	$\frac{\sqrt{3}+1}{2\sqrt{2}}$
1.	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$
.799	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$
.744	$\frac{\pi}{3}$	$\frac{1}{2}$
.799	$\frac{5\pi}{12}$	$\frac{\sqrt{3}-1}{2\sqrt{2}}$
1.	$\frac{\pi}{2}$	0

To find the curve of intersection of the surface and Z = 1/2. Put this value in equation of surface and we get

$$\rho^6+\rho^3(6\cos\theta - 8\cos^3\theta)+1/2 = 0$$

$$\rho^3 = \frac{-(6\cos\theta - 8\cos^3\theta) \pm\sqrt{(6\cos\theta - 8\cos^3\theta)^2 - 2}}{2}$$

Calculating values we get

$\cos\theta$	θ	ρ	ρ
1	0	1.19	.29
$\frac{\sqrt{3}+1}{2\sqrt{2}}$	$\frac{\pi}{12}$	.8921	
$\frac{\sqrt{3}}{2}$	$\frac{\pi}{6}$	-----	
$\frac{1}{\sqrt{2}}$	$\frac{\pi}{4}$	.8921	
$\frac{1}{2}$	$\frac{\pi}{3}$	- .66	
$\frac{\sqrt{3}-1}{2\sqrt{2}}$	$\frac{5\pi}{12}$	-.89	
0	$\frac{\pi}{2}$	-------	

Let us consider now the conformal mapping of the function, $W = z^3 - 1$, from the W plane to the z plane. Let $|W| = 0$. This gives the origin in the W plane and makes $z^3 = 1$, $z = 1, \omega, \omega^2$. Thus the origin maps into the three points. If $|W|^2 = 1$, then we have a circle of unit radius about the origin in the W plane and the three loops as shown in figure 22 in the z plane. For $|W|^2 = 1/2$ we get another circle, concentric with the first two in the W plane and having a radius equal to $\sqrt{\frac{1}{2}}$, and the three ovals in the z plane. For $|W|^2 = k$, the general case, we get a circle about the origin in the W plane, with a radius equal to $\sqrt{k}$ and in the z plane a figure such as is shown by the outermost curve of Fig .

Since the pencil of circlesmaps into these curves, their orthogonal trajectories map into the orthogonal trajectories of the second set. The lines $u = mv$ are the orthogonal trajectories of the circles in the W plane. But from the equation $u = x^3 - 3xy^2 - 1$ and $V = 3x^2y - y^3$ so that $(x^3 - 3xy^2 - 1) = m(3x^2y - y^3)$ is the equation of the orthogonal trajectories in the xy plane, Fig.

Let m = 0. Then we get

$m = 0$

x	y
0	∞
1	0
1/2	Imag.
2	$\sqrt{7/6} = \pm 1.07$
3/2	$\pm .7265$
3	± 1.7

$m = \infty$

x	y
± 1	3.
± 2	12.
± 3	27.
$\pm 1/2$	3/4
$\pm 1/4$	3/16
$\pm 3/2$	6.75

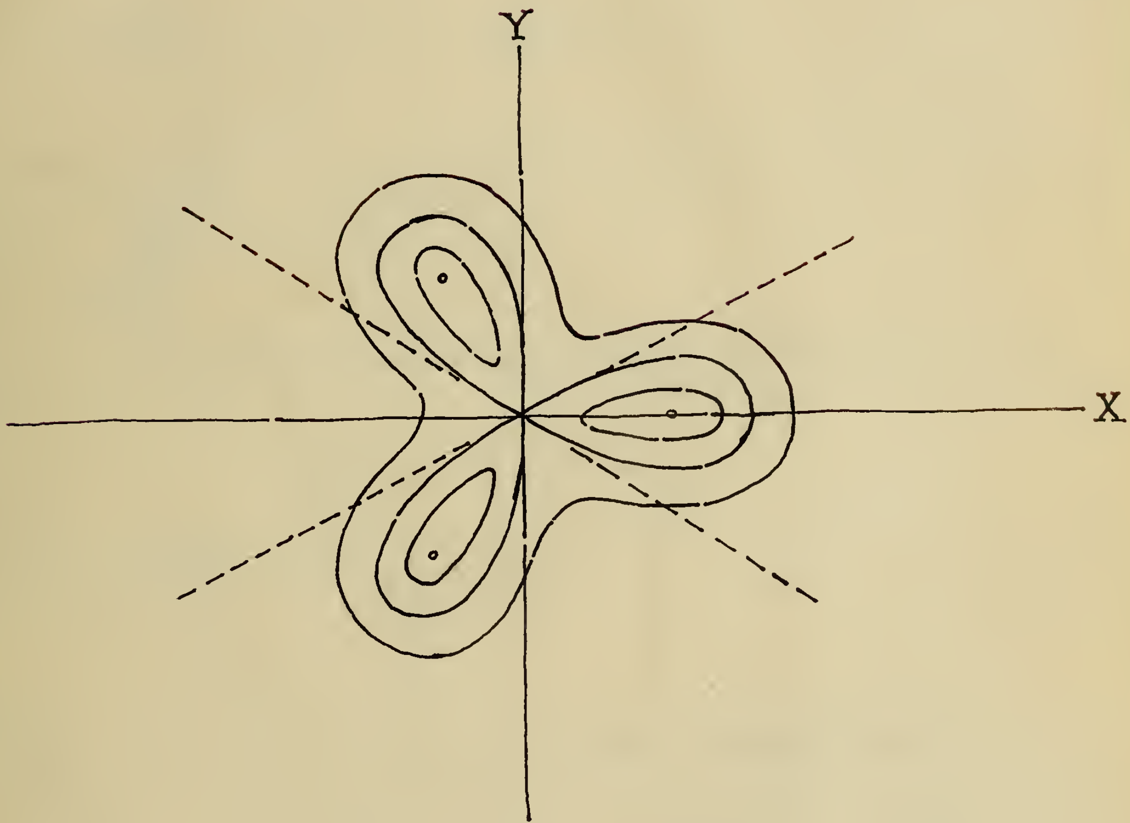
$$3x^2y - y^3 = 0$$

$$y^2(3x^2 - y) = 0$$

$$y = 0$$

$$3x^2 = y$$

FIG. XXI



$$(x^2+y^2)^3 - 2x^3 + 1 + \text{const.} = 0$$

So far the general case $W = z^n - 1$; if $W = 1$, $z^n = 0$ and we have n zero points of the function. Z or $u^2 + v^2$ will be expressed by an equation of degree 2n in x and y. Expressing this equation in polar coordinates we have

$\rho^n(\cos n\theta - i\sin n\theta) = W - 1$

$u = \rho^n \cos n\theta - 1$

$v = \rho^n \sin n\theta$

$Z = u^2 + v^2 = \rho^{2n}(\cos^2 n\theta + \sin^2 n\theta) - 2\rho^n \cos n\theta + 1$

$u^2 + v^2 - 1 = \rho^{2n} - 2\rho^n \cos n\theta.$

The equation of the curve of intersection of this surface and the plane Z = 1 is therefore

$\rho^{2n} - 2\rho^n \cos n\theta = 0$

$\rho^n = 2\cos n\theta$. Expressing this in rectangular coordinates we have

$(x^2 + y^2)^n = x^{2n} + n x^{2n-2} y^2 + \text{-----}$

For $\cos n\theta =$

$\sin\theta = \dfrac{y}{\sqrt{x^2 + y^2}}$ and $\cos\theta = \dfrac{x}{\sqrt{x^2 + y^2}}$

VI.

So far we have considered only algebraic functions. The investigation may, of course, be extended to any regular analytic function. I shall confine myself to a simple example of this kind and merely discuss the shape of the surface.

(1) $W = e^z$

$W = e^{x+iy} = e^x \cdot e^{iy}$

$= e^x \cos y + ie^x \sin y$

$Z = u^2 + v^2 = e^{2x}\cos^2 y + e^{2x}\sin^2 y + 2e^{2x}\sin y \cos y$ A. E.

(2) $Z = e^{2x}(1 + \cancel{\sin 2y})$ which gives us the equation of the surface.

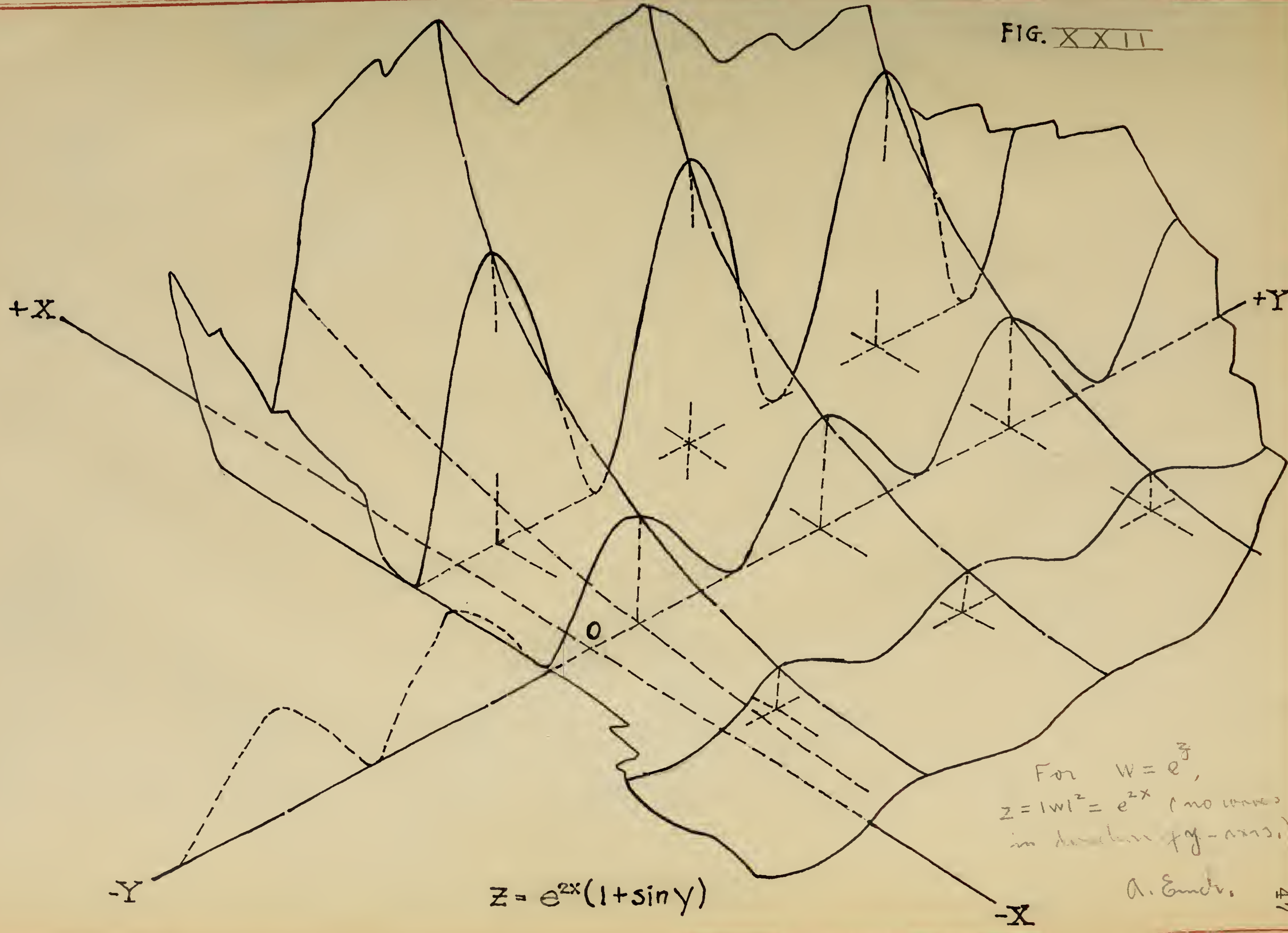
To find the intersection of the surface and the xz plane put $y = 0$ and we have $z = e^{2x}$, which gives the following table of values.

x	z
0	1
1	7.29
2	52.
3	384.
1/2	2.7

To find the intersection of the surface and the yz plane, put $x = 0$ and we get $Z = 1 - \sin 2y$, which gives the forrowing table of values.

z	y
1	0
2	$\pi/4$
1	$\pi/2$
0	$3\pi/4$
1	π

FIG. XXII



$Z = e^{2x}(1+\sin y)$

For $w = e^{z}$,
$z = |w|^2 = e^{2x}$ (no waves in direction of y-axis.)

A. Emch.